Percolation: Theory and Applications

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Content:

1. **Percolation Theory:** phase transition, critical exponents, geometrical properties, substructures (backbone, red-bond, shortest path), universality, critical dimension, directed percolation, anomalous transport.

2. **Applications:** Optimal path, directed polymers, epidemics, immunization, oil recovery, nanomagnets, etc.

3. **Fractals:** Fractals in Nature, mathematical fractals, self-similarity, scaling laws, relation to chaos, multifractals.

4. **Networks:** classical networks, Erdos Renyi graphs, small world, scale free, Internet and www, biological networks, social networks, models for epidemic spreading.

Books

1. A. Bunde and S. Havlin (eds): Fractals and Disordered Systems (2nd Ed, Springer, Berlin 1996); Fractals in Science (Springer, Berlin 1994).

2. D. Stauffer and A. Aharony: Introduction to Percolation (1992).

3. S. Havlin and D. Ben Avraham, Diffusion in Random Media, Adv. in Phys. 36, 659 (1987).

4. D. ben Avraham and S. Havlin, Diffusion and Reactions in Fractals and Disordered Systems (Cambridge University press 2000)

5. B.B. Mandelbrot: The Fractal Geometry of Nature (Freeman, San Francisco 1982).

6. T. Vicsek: Fractal Growth Phenomena (World Scientific, Singapore 1992).

7. J. Feder: Fractals (Plenum, NY 1988).

8. H.O. Peitgen, H. Jurgens and D. Saupe: Chaos and Fractals (Springer, NY 1992).

9. P. Bak, How Nature Works (Copernicus, NY 1996).

10. James Gleick, Chaos (Penguin books, NY 1997).

11. P. Meakin, Fractals, Scaling and Growth far from Equilibrium (Cambridge University press, 1998).

12. A. L. Barabasi, Linked (Plume books, 2003).

13. R. Pastor-Satorras, A. Vespignani, Evolution and Structure of the Internet: A Statistical Physics Approach (Cambridge University Press, 2004).

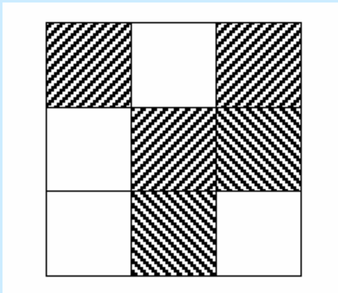
14. S. N. Dorogovtsev, J. F. F. Mendes, Evolution of Networks: From Biological Nets to the Internet and www (Physics) (Oxford University Press, 2003).

15. R. Cohen and S. Havlin, Complex Networks: Structure, Robustness and Function (Oxford University Press, 2010).

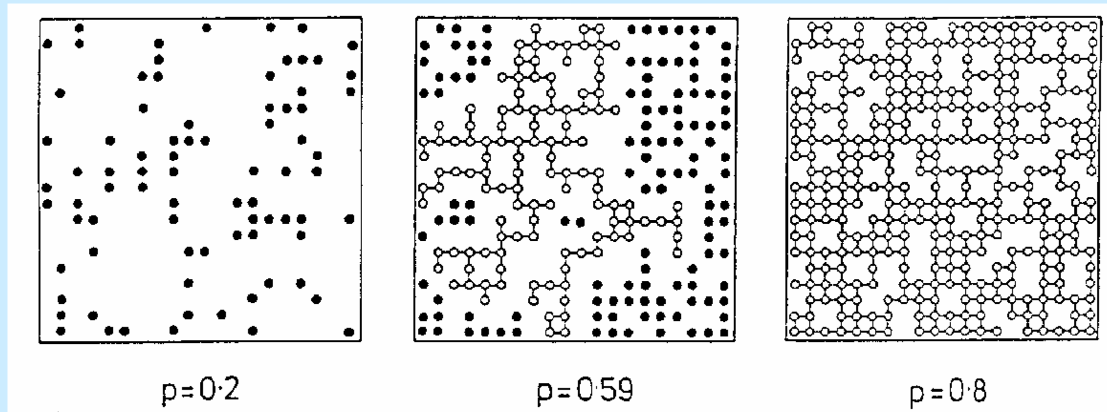
Percolation

- Model for disordered media

P=1/2



- Each site is occupied with probability p and empty with probability 1-p



- For low p – small clusters
- For large p – big clusters – Infinite cluster
- At $p=p_c$ a transition from small clusters to infinite clusters
- Occupied and empty sites can represent different physical properties, e.g.
  occupied – conductors
  empty – isolators
- Current can flow only on conductors

$$\left.\begin{array}{l}\text{below } p_c \text{ – isolator} \\ \text{above } p_c \text{ – conductor}\end{array}\right\} \text{Isolator-conductor phase transition}$$

- p_c – called “critical concentration” – above which current cannot flow
- p_c – called also “percolation threshold”

Percolation

More examples

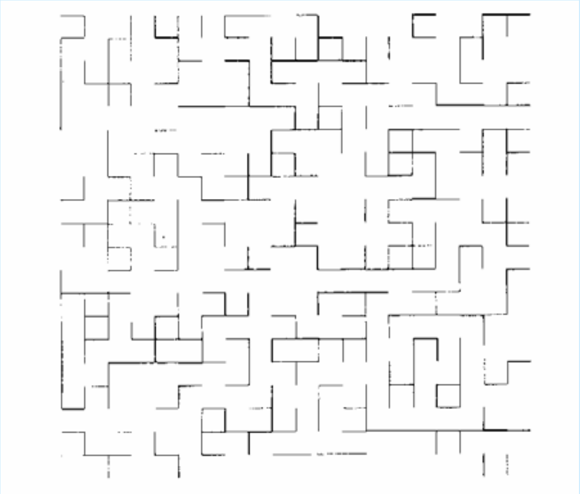
✓Occupied sites – superconductors
Empty sites - conductors
} Superconductor – conductor phase transition (at p_c)

✓Occupied sites – magnets
Empty sites - paramagnets
} Magnet - paramagnet phase transition (at p_c)

✓Occupied sites – working computers
Empty sites – damaged computers
} Internet network phase transition

✓Comparison with thermal phase transition

solid-liquid
critical temperature T_c
below T_c – order (infinite cluster)
above T_c – disorder (small clusters)

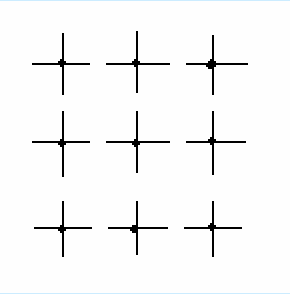


Bond Percolation

✓Bonds are occupied randomly with probability p
✓At p_c an infinite cluster of bonds appears
✓Model for random resistor network: bonds are cut randomly

Bond Percolation - Examples

Chemistry - polymerization



- ✓Branching molecules can perform larger molecules by activating more and more bonds
- ✓Assume that probability to activate a bond is p

  below p_c – small macromolecules

  above p_c – large macromolecules (system size)
- ✓Called sol-gel transition

Gel – infinite cluster – elastic (like food gels) – above p_c

Sol – viscous fluid – below p_c

- ✓Example – boiled egg

  heating – activates more bonds between molecules

Biology – epidemic spreading

- ✓Epidemic starts with a single sick person that can infect its neighbors with probability p (per unit time)
- ✓Neighbors can infect their neighbors
- ✓If p is small the epidemic stops. Above p_c the epidemic spreads to large populations
- ✓Model also for fire spreading in a forest

✓Percolation aspects are important in many systems in Nature: amorphous and porous materials (e.g. rocks), branched polymers, fragmentation, galaxies structure, earthquakes, anomalous properties of water, network such the Internet, immunization, optimization, minimal spanning trees, simulations of oil recovery from porous rocks.

Percolation Threshold

✓Site and bond percolation can be defined for all lattices and for all d

✓In general a bond has more neighbors than a site

Example: square lattice site has 4 neighbors

bond has 6 neighbors

Thus, big clusters of bonds are easier generated than for sites

$$\Rightarrow p_c \text{ for bonds} < p_c \text{ for sites}$$ for the same lattice

✓Example: $p_c = 1/2$ for bond percolation, $p_c = 0.593$ for site percolation } on square lattice

Percolation		Lattice
bond - p_c	**site -** p_c	
$2\sin \pi/18$	$1/2$	Triangle
$1/2$	0.5927	Square
0.2488	0.3116	Cubic

Continuum Percolation

- ✓Natural example – continuum percolation
- ✓Two components not on a lattice
- ✓Example: take a conducting plate
  make circular holes randomly

- ✓Called: Swiss Cheese Model
- ✓P_c=0.312±0.005 for d=2; p_c=0.034 for d=3
  above p_c – conductor
  below p_c – insulator
- ✓Model for porous materials

Historical remarks

- ✓ First work on percolation – Flory +Stockmayer (1941-1943)
  studied gelation or polymerization

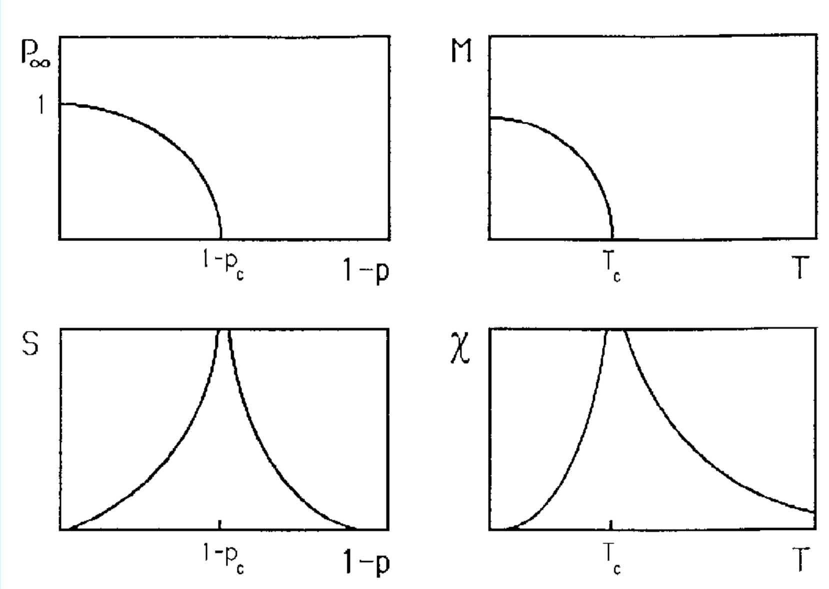
- ✓ Name percolation – Broadbent and Hammersley (1957)
  studied flow of liquid in porous media
  presented several concepts in percolation

- ✓ The developments in phase transition (1960's), series expansion (Domb), renormalization group, scaling theory and universality by Wilson (Nobel Prize), Fisher and Kadanoff – helped to develop percolation theory and understand the percolation as a critical phenomena

- ✓ Fractal concept (Mandelbrot, 1977) – new tools (fractal geometry) together with computer development ⇒ pushed forward the percolation theory

- ✓ Still – many open questions exist !

Percolation – Phase Transition



- ✓Example of a geometrical phase transition
- ✓p_c – critical threshold separates two phases:
  - (1) ordered $p>p_c$ – infinite cluster
  - (2) disordered $p<p_c$ – finite clusters
- ✓Analogy to { thermodynamic phase transition; magnetic phase transition }

Ferromagnetic – paramagnetic phase transition

$T<T_c$ spontaneous magnetization M>0 – ferromagnetic phase
interaction between spins $\Rightarrow$ order

$T>T_c$ no magnetization M=0 – paramagnetic phase
termal energy $\Rightarrow$ disorder

M – called "order parameter" scales as $M \sim (T_c - T)^{\beta}$

χ - magnetic fluctuations – susceptibility

$$\chi \sim \left\langle (M - \bar{M})^2 \right\rangle^{1/2} \sim |T - T_c|^{-\gamma}$$

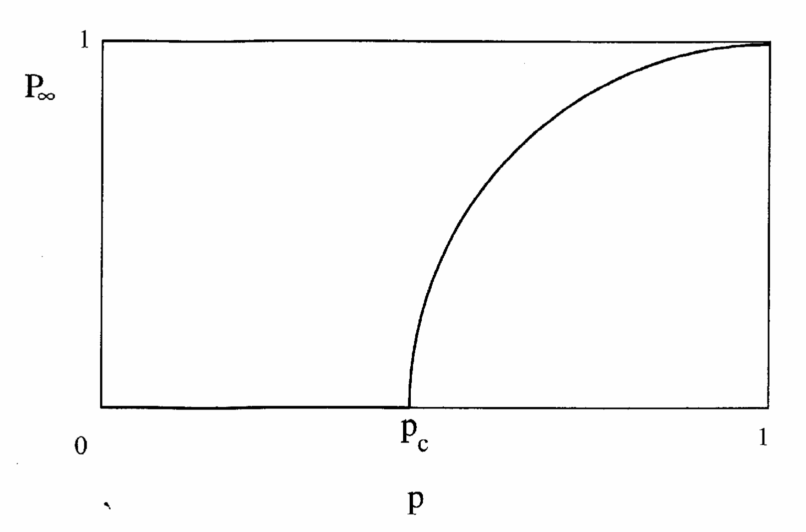
ξ - correlation length (size of ordered clusters)

$$\xi \sim |T_c - T|^{-\nu}$$

β, γ, ν - called critical exponents

Percolation – critical exponent

✓p – same role as T in thermal phase transitions
✓P_∞ - probability that a site (bond) belongs to ∞ cluster
order parameter $P_\infty \propto (p-p_c)^\beta$- similar to magnetization



✓ ξ - correlation length – mean distance between two sites on the same finite cluster

$$\xi \propto |p-p_c|^{-\nu}$$

✓The average size of finite clusters $S \sim |p-p_c|^{-\gamma}$
(analogous to susceptibility)

✓ ν and γ are the same for $p>p_c$ and $p<p_c$
✓For ξ and S take into account all finite clusters
✓ β,ν and γ called critical exponents $\Rightarrow$ describe critical behavior near the transition
✓The exponents are universal
✓Universality – property of second order phase transition (order parameter →0 continuously)
All magnets in d=3 have same β
independent on the lattice and type of interactions
✓T_c – depends on details (interactions, lattice) – same for p_c

Percolation	$d=2$	$d=3$	$d\geq 6$
Order parameter $P_\infty : \beta$	$5/36$	0.417 ± 0.003	1
Correlation length $\xi : v$	$4/3$	0.875 ± 0.008	$1/2$
Mean cluster size $S : \gamma$	$43/18$	1.795 ± 0.005	1
Magnetism	$d=2$	$d=3$	$d\geq 4$
Order parameter $m : \beta$	$1/8$	0.32	$1/2$
Correlation length $\xi : v$	1	0.63	$1/2$
Susceptibility $X : \gamma$	$7/4$	1.24	1