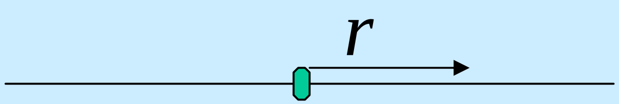
Exact Results

Only few exact results exist!

a) One dimensional systems:



Since infinite cluster can occur only if all sites are occupied

$$p_c = 1$$

Thus only quantities below p_c such as correlation length ξ and mean size of the clusters S are relevant!

The correlation function $g(r)$, defined as the prob. to have at distance r a site on the same cluster.

$$g(r) = 2p^r$$

The correlation length ξ is defined as the mean distance between two sites on the same cluster

$$\xi^2 = \frac{\sum_{r=1}^{\infty} r^2 g(r)}{\sum_{r=1}^{\infty} g(r)} = \frac{\sum_{r=1}^{\infty} r^2 p^r}{\sum_{r=1}^{\infty} p^r}$$

The sums can be performed easily

$$\xi^2 = \frac{1+p}{(1-p)^2} = \frac{1+p}{(p_c-p)^2} \quad \Rightarrow \quad \xi = \frac{\sqrt{1+p}}{(p_c-p)}$$

Thus $\nu = 1$ in one dimension. The correlation function $g(r)$ near p_c

$$g(r) \sim e^{-r/\xi},$$

where the correlation length ξ represents the decay radius of the correlation function.

The mean mass S of the finite clusters is

$$S = 1 + \sum_{r=1}^{\infty} g(r) = \frac{1+p}{1-p} \sim (p_c - p)^{-1}.$$

The 1 comes from the site at the origin, which was assumed to be occupied

Hence $\gamma = 1$ in one dimension.

The probability that a chosen lattice site belongs to a cluster of s sites is $sp^s(1-p)^2$. The factor s is due to the fact that the chosen site can be any of the s sites in the cluster. The factor $(1-p)^2$ is due to the fact that every cluster must be surrounded by perimeter sites which are empty. In *d*=1, every cluster has two perimeter sites. The corresponding probability per cluster site, , is defined

$$n_s = p^s(1-p)^2$$

n_s - probability that a cluster is of size s. n_s is also the number of clusters of size s divided by the total number of sites in the system. Thus, $\sum_{s=1}^{\infty} sn_s = p.$

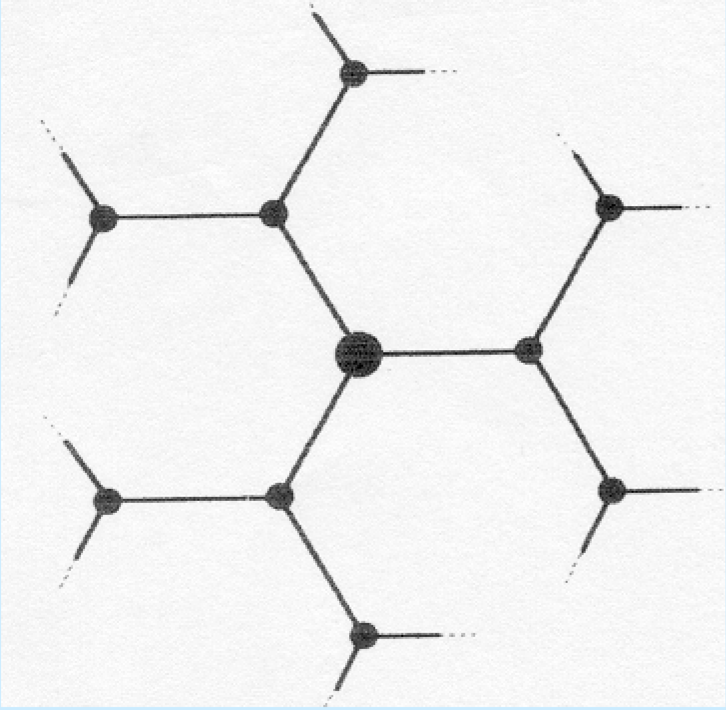
The mean cluster mass S is related to n_s by

$$S = \sum_{s=1}^{\infty} s\left(\frac{sn_s}{\sum_{s=1}^{\infty} sn_s}\right) = \frac{1+p}{1-p} \sim (p_c - p)^{-1}.$$

The factor $(sn_s / \sum sn_s)$ is the probability that an occupied site belongs to a cluster of s sites.

The Cayley Tree

The Cayley tree is a structure without loops. From each site $z-1$ new branches grow out, generating $z(z-1)$ sites in the second shell. For $z=2$, the tree reduces to the one-dimensional chain.



Two shells of a Cayley tree, with $z=3$

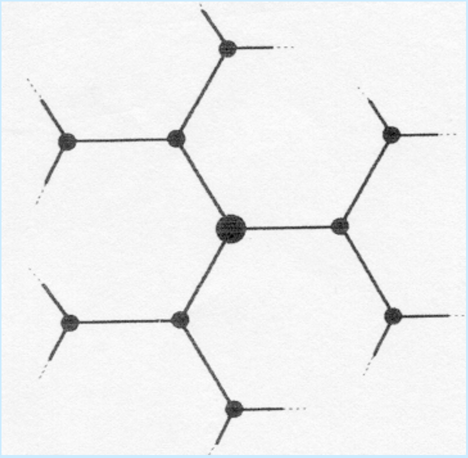
- There are no loops in the system, since any two sites are connected by *only one* path.

- The Euclidean distance *r* has no meaning.

- The lattice is described solely by the (shortest) chemical distance ℓ between two sites.

- For example, the chemical distance between the central site and a site on the ℓth shell is exactly ℓ .

- The ℓth shell of the tree consists of $z(z-1)^{\ell-1}$ sites, increasing exponentially with ℓ .



- In a *d*-dimensional Euclidean lattice, with *d* finite, the number of sites at distance ℓ increases as ℓ^{d-1}

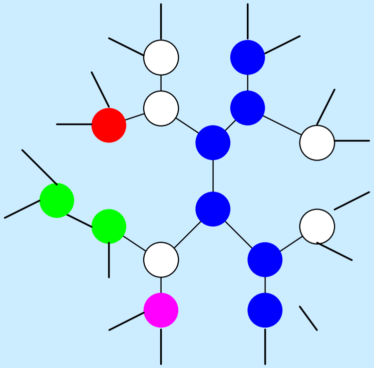
- Since the exponential dependence can be considered as a power-law behavior with an infinite d (dimension), the Cayley tree can be regarded as an infinite-dimensional lattice.

- From the universality property we can expect that the critical exponents derived for percolation on the Cayley tree will be the same as for percolation on *any* infinite-dimensional lattice.

- It is known that the upper critical dimension for percolation is d_c=6, i.e., for $d \geq 6$ the critical exponents are the same for all dimensions.

- Thus we expect that the exponents for percolation on the Cayley tree are the same as in $d \geq 6$ dimensions.

Percolation on a Cayley Tree



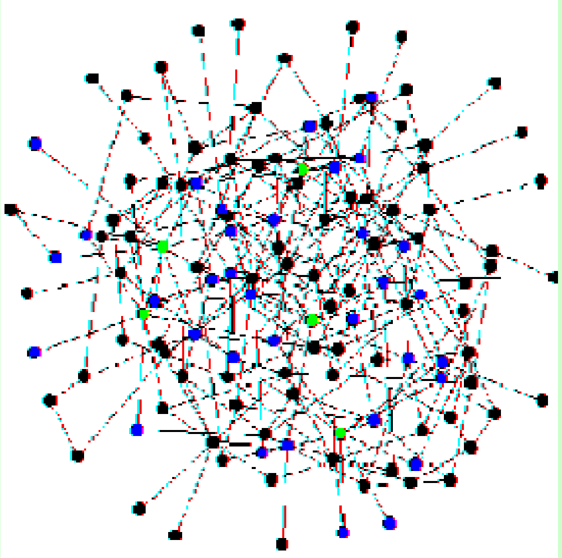
- **Contains no loops**
- **Connectivity of each node is fixed (z connections)**
- **Critical threshold:**

$$p_c = \frac{1}{z-1}$$

- **Behavior of the spanning cluster size near the transition is linear:**

$$P_\infty \propto (p - p_c)^\beta, \quad \beta = 1$$

Random Graph Theory



- **Developed in the 1960's by Erdos and Renyi.** **(Publications of the Mathematical Institute of the Hungarian Academy of Sciences, 1960).**

- **Discusses the ensemble of graphs with N vertices and M edges (2M links).**
- **Distribution of connectivity per vertex is Poissonian (exponential), where k is the number of links :**

$$P(k) = e^{-c}\frac{c^k}{k!}, \qquad c = \langle k \rangle = \frac{2M}{N}$$

- **Distance d=log N -- SMALL WORLD**

More Results

- **Phase transition at average connectivity, $\langle k \rangle = 1$:**

  $\langle k \rangle < 1$ **No spanning cluster (giant component) of order** $logN$

  $\langle k \rangle > 1$ **A spanning cluster exists (unique) of order** N

  $\langle k \rangle = 1$ **The largest cluster is of order** $N^{2/3}$

- **Size of the spanning cluster is determined by the self-consistent equation:**

$$P_\infty = 1 - e^{-\langle k \rangle P_\infty}$$

- **Behavior of the spanning cluster size near the transition is linear:**

$P_\infty \propto (p_c - p)^\beta$, $\beta = 1$, **where** p **is the probability of deleting a site,**

$p_c = 1 - 1/\langle k \rangle$