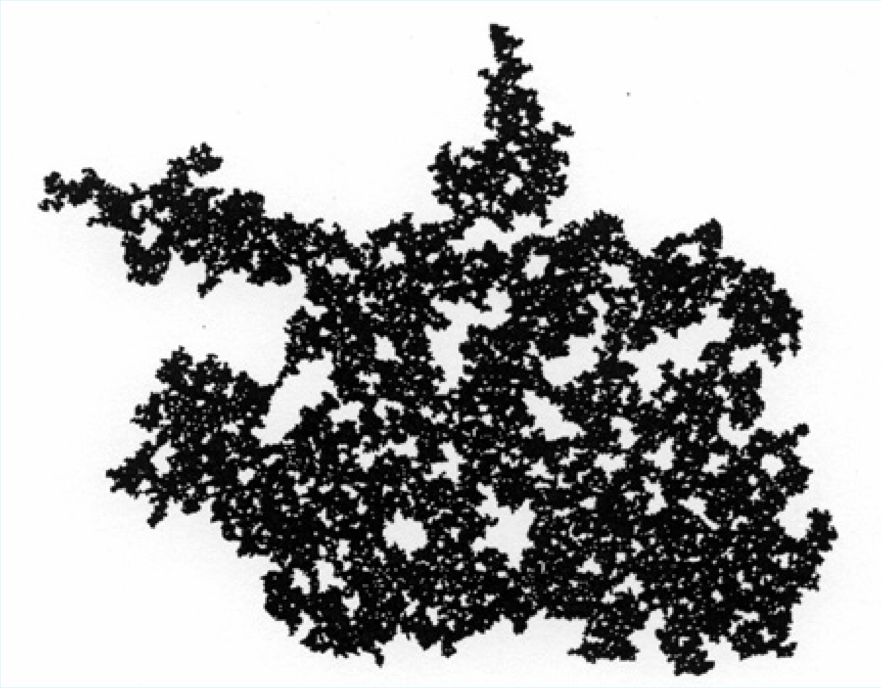
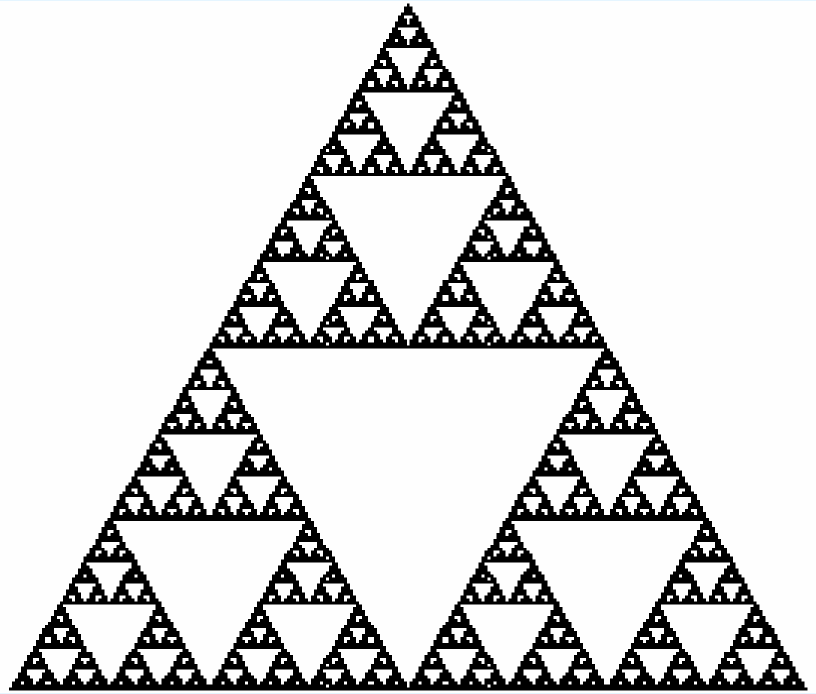
Percolation – Geometrical Properties

- A percolation cluster can be characterized by fractal geometry
- We can see in the infinite cluster, at p_c, holes in all scales – like Sierpinski gasket
- The cluster is self-similar (from pixel size to system size)





Fractals

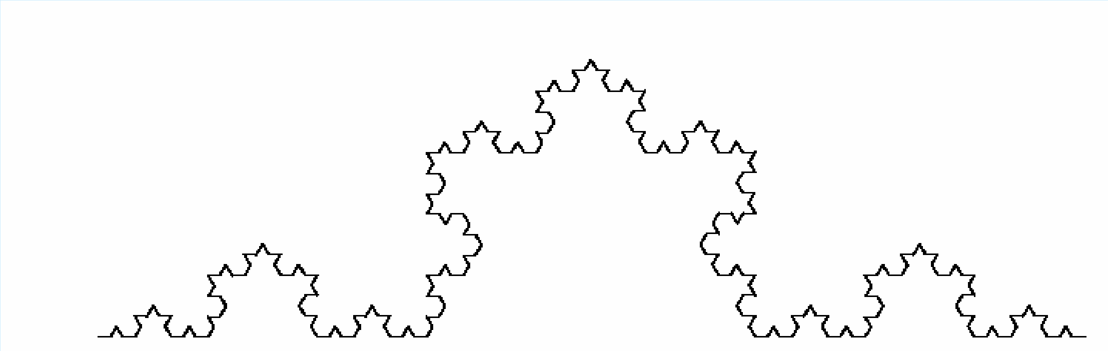
Fractal geometry describes Nature better than classical geometry.
Two types of fractals: deterministic and random.

Deterministic fractals

Ideal fractals having self-similarity.
Every small part of the picture when magnified properly, is the same as the whole picture.

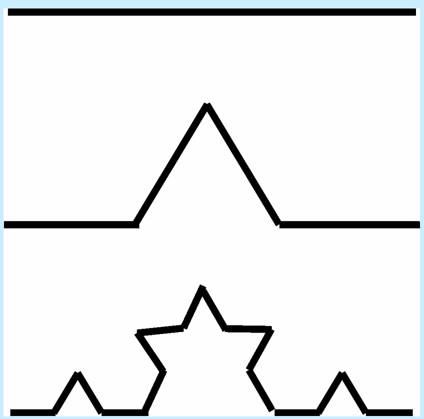
Self-similarity is a property, not a definition

To better understand fractals, we discuss several examples:



Koch curve

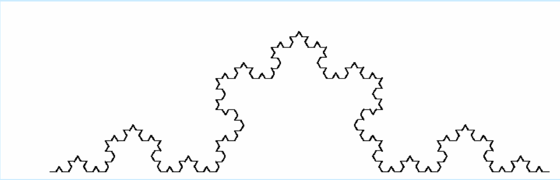
Building Koch curve



$n=0$ A section of unit length

$n=1$ Divide each section to 3 equal pieces and replace the middle one by two pieces like a tent

$n=2$ The same is done for all 4 sections



$n=\infty$

This is a mathematical fractal

In physics we continue until $n_{\max}$. We have a fractal for length scales $1/3^{n_{\max}} < x < 1$

Koch curve properties:

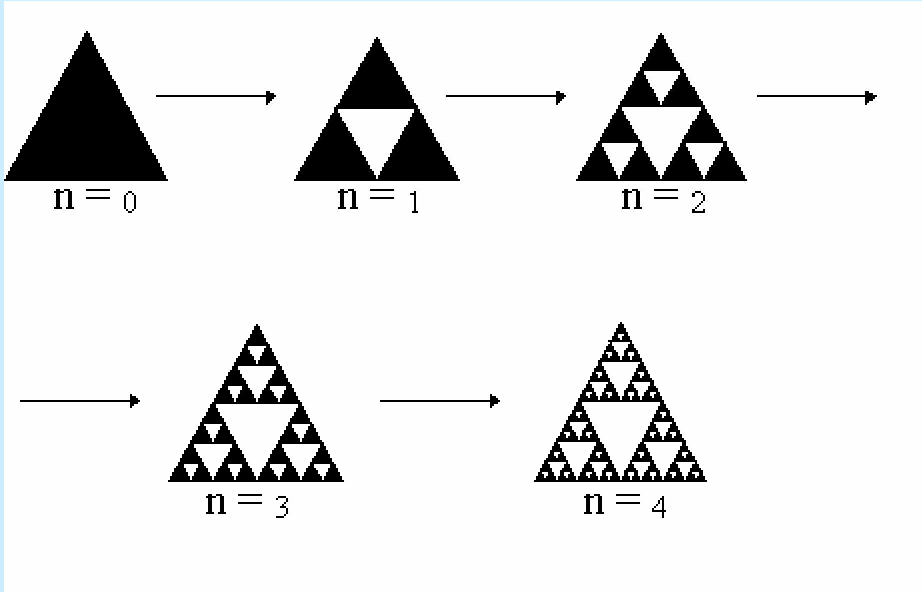
(a) $\left(\frac{4}{3}\right)^n = Length \to \infty \quad for \quad n = \infty$. But contains in a finite space. No derivative.

(b) Self-similarity – scale invariance

(c) No characteristic scale

Sierpinski gasket is perhaps the most popular fractal.

Generation of Sierpinski gasket

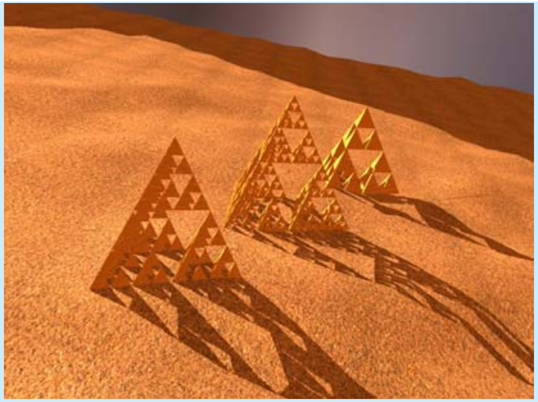


(1) divide an equilateral triangle into 4 equal triangles
(2) take out the central one
(3) repeat this for every triangle

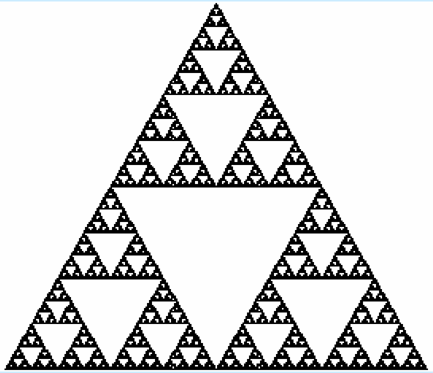
No translation symmetry
Scale invariance symmetry

Internal perimeter: $\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\ldots\rightarrow\infty$

Area: $S_0, \frac{3}{4}S_0, \left(\frac{3}{4}\right)^2 S_0 \ldots \rightarrow 0$

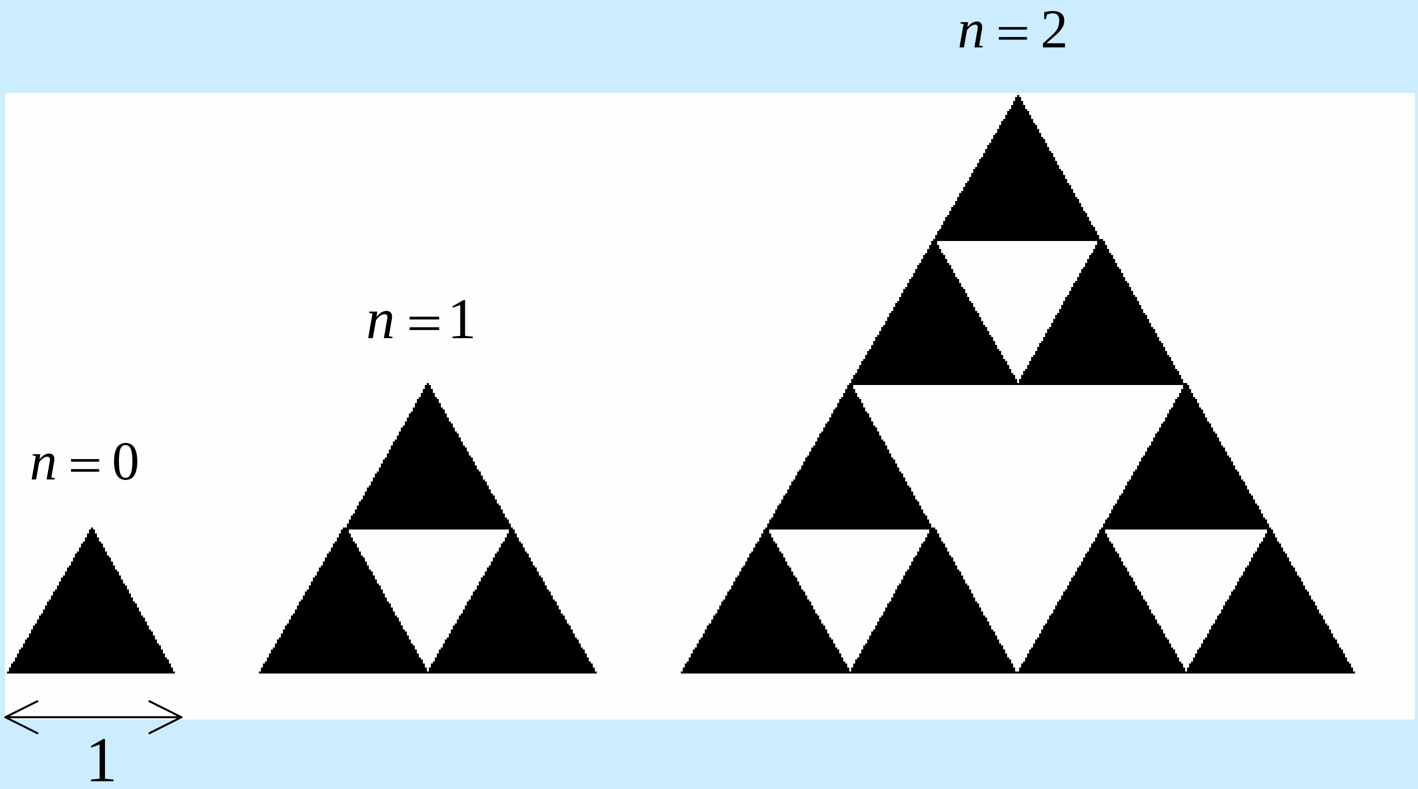


3D Sierpinski gasket



2D Sierpinski gasket

Sierpisnki gasket with lower cut off

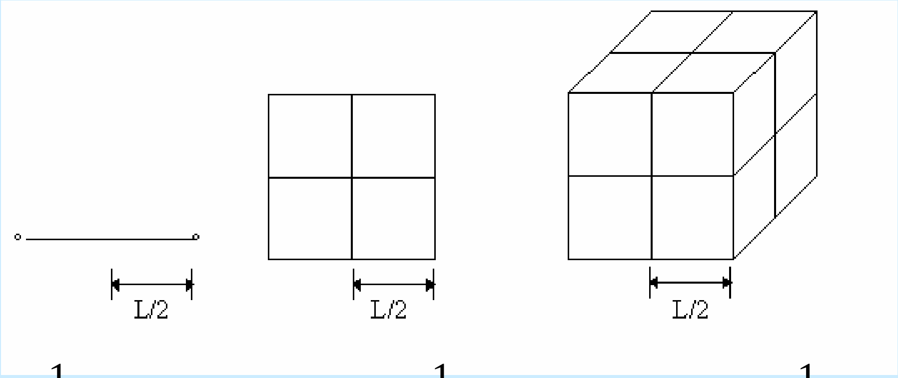


This is a fractal for $1 < x < 3^{n_{\max}}$

Fractal dimension

How to quantify fractals ?
Generalization of dimension to non-integer dimensions – fractal dimension (B.B. Mandelbrot, 1977)

Definition of dimension



$$M(L/2) = \frac{1}{2} M(L) \qquad M(L/2) = \frac{1}{4} M(L) \qquad M(L/2) = \frac{1}{8} M(L)$$

$$d = 1 \qquad d = 2 \qquad d = 3$$

- Take a line section of length L, divide into two, we get: $M\left(\frac{1}{2}L\right) = \frac{1}{2} M(L)$
- Take a square of length L, divide L by 2 we get: $M\left(\frac{1}{2}L\right) = \frac{1}{4} M(L) = \frac{1}{2^2} M(L)$
- Take a qube of length L, divide L by 2 we get: $M\left(\frac{1}{2}L\right) = \frac{1}{8} M(L) = \frac{1}{2^3} M(L)$

In general

$$M(bL) = b^d M(L)$$

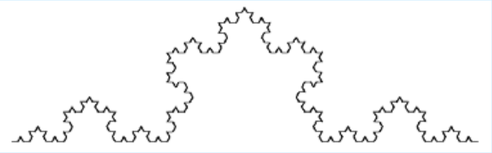
The exponent d defines the dimension of system

Solution: $M(L) = AL^d$ where A is a constant

Definition of fractal dimension $M(bL) = b^{d_f} M(L)$

generalization to non-integer dimension d_f

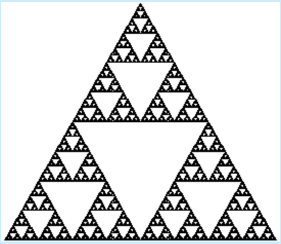
Solution: $M(L) = AL^{d_f}$



Example: Koch curve

$$M\left(\frac{1}{3}L\right) = \frac{1}{4}M(L) = \left(\frac{1}{3}\right)^{d_f} M(L) \Rightarrow \left(\frac{1}{3}\right)^{d_f} = \frac{1}{4} \quad or \quad d_f = \frac{\log 4}{\log 3} \approx 1.262$$

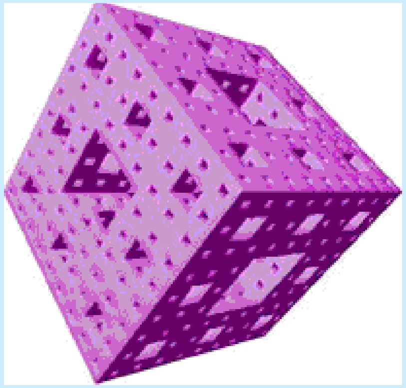
d_f - non integer – between 1 and 2 dimensions. Koch curve is not a line (d=1) but doesn't fill a plane (d=2).



Example: Sierpinski gasket

$$M\left(\frac{1}{2}L\right) = \frac{1}{3}M(L) = \left(\frac{1}{2}\right)^{d_f} M(L) \Rightarrow \left(\frac{1}{2}\right)^{d_f} = \frac{1}{3} \quad or \quad d_f = \frac{\log 3}{\log 2} \approx 1.585$$

Non integer dimension between 1 and 2 dimensions.



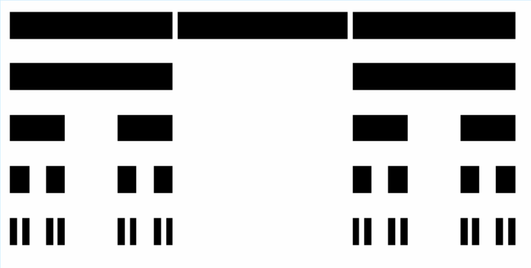
Example: Sierpinski sponge:

$$M\left(\frac{1}{3}L\right)=\frac{1}{20}M(L)=\left(\frac{1}{3}\right)^{d_f}M(L)\Rightarrow\left(\frac{1}{3}\right)^{d_f}=\frac{1}{20}\quad or\quad d_f=\frac{\log 20}{\log 3}\approx 2.727$$

Here the fractal dimension is between 2 and 3.

Are there fractals with $d_f<1$?

Example: Cantor set

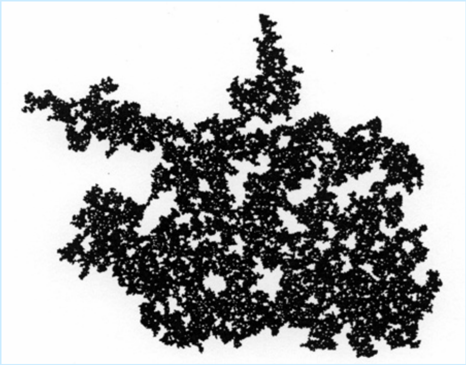


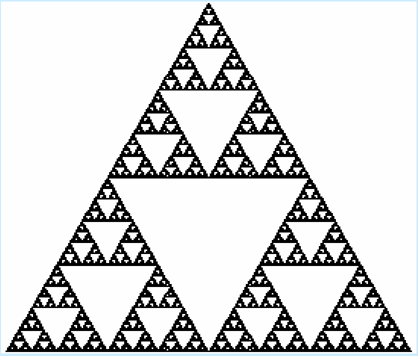
A section of unit size.
Divide into 3 equal sections and remove the central one.
Repeat it for every left section.
For $n\to\infty$ we get a fractal set of points.

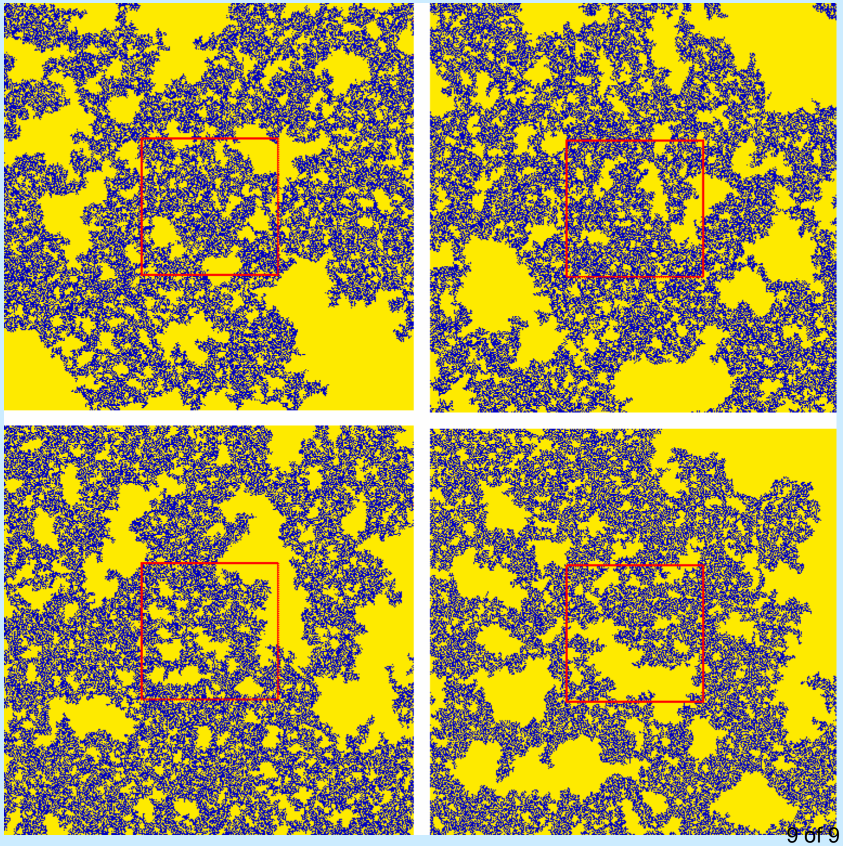
$$M\left(\frac{1}{3}L\right)=\frac{1}{2}M(L)=\left(\frac{1}{3}\right)^{d_f}M(L)\Rightarrow\left(\frac{1}{3}\right)^{d_f}=\frac{1}{2}\quad or\quad d_f=\frac{\log 2}{\log 3}\approx 0.631$$

Percolation – Geometrical Properties

- A percolation cluster can be characterized by fractal geometry
- We can see in the infinite cluster, at p_c, holes in all scales – like Sierpinski gasket
- The cluster is self-similar (from pixel size to system size)







- The square in left top is magnified in right top
  $\Rightarrow$ magnified in left bottom
  $\Rightarrow$ magnified in right bottom
- The difficulty to easily realize the order is a sign of self-similarity