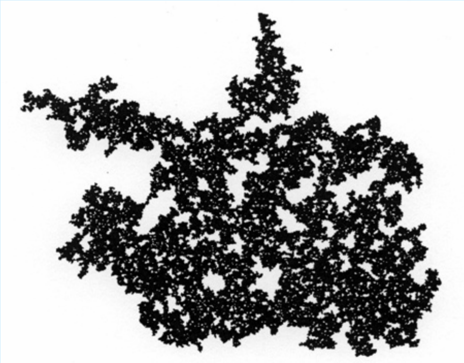
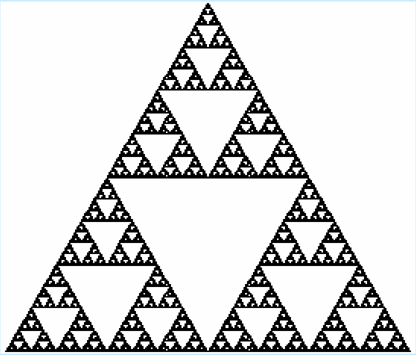
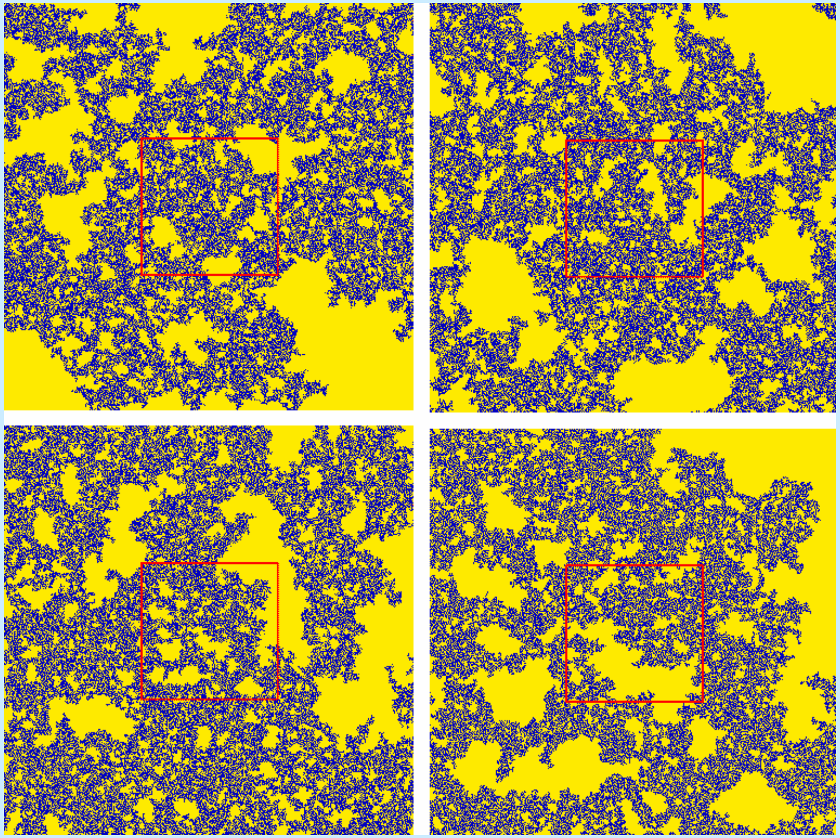
Percolation – Geometrical Properties

- A percolation cluster can be characterized by fractal geometry
- We can see in the infinite cluster, at p_c, holes in all scales – like Sierpinski gasket
- The cluster is self-similar (from pixel size to system size)





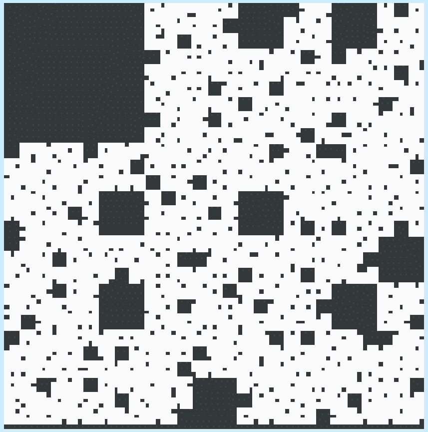


- The square in left top is magnified in right top
  ⇒ magnified in left bottom
  ⇒ magnified in right bottom
- The difficulty to easily realize the order is a sign of self-similarity

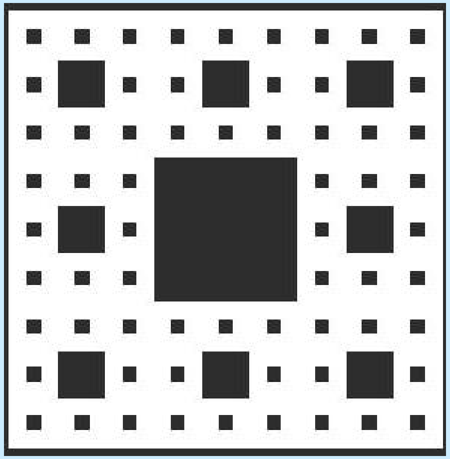
Random Fractals

- Fractals in nature are not deterministic
- One can generate random fractals
- Instead of always removing the central square, we remove randomly one of the 9 squares

Random Sierpinski carpet



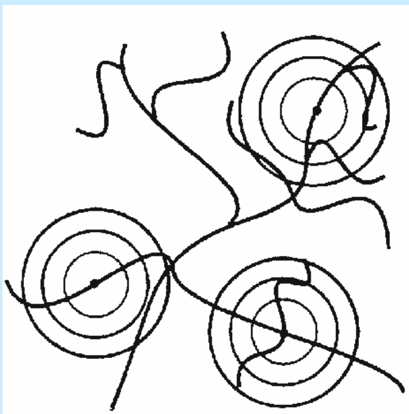
Deterministic Sierpinski carpet



- The fractal dimension of the random Sierpinski carpet is the same as the deterministic: $M\left(\frac{1}{3}L\right)=\frac{1}{8}M(L)=\left(\frac{1}{3}\right)^{d_f}M(L), \quad d_f=\frac{\log 8}{\log 3}\cong 1.893$
- The self-similarity is not exact – valid statistically

Random Fractals – Fractal Dimension

Methods: (a) cluster growing; (b) box counting; (c) correlations.



- Choose a site on the fractal – origin
- plot circles of several radiuses $r \ll R_{\max}$
- $R_{\max} \approx$ radius of the fractal r
- count the number of sites inside
- repeat the measurements for several origins
- average over all $M(r)$ for each r
- plot $M(r)$ vs r on log-log plot
- the slope is d_f of the fractal

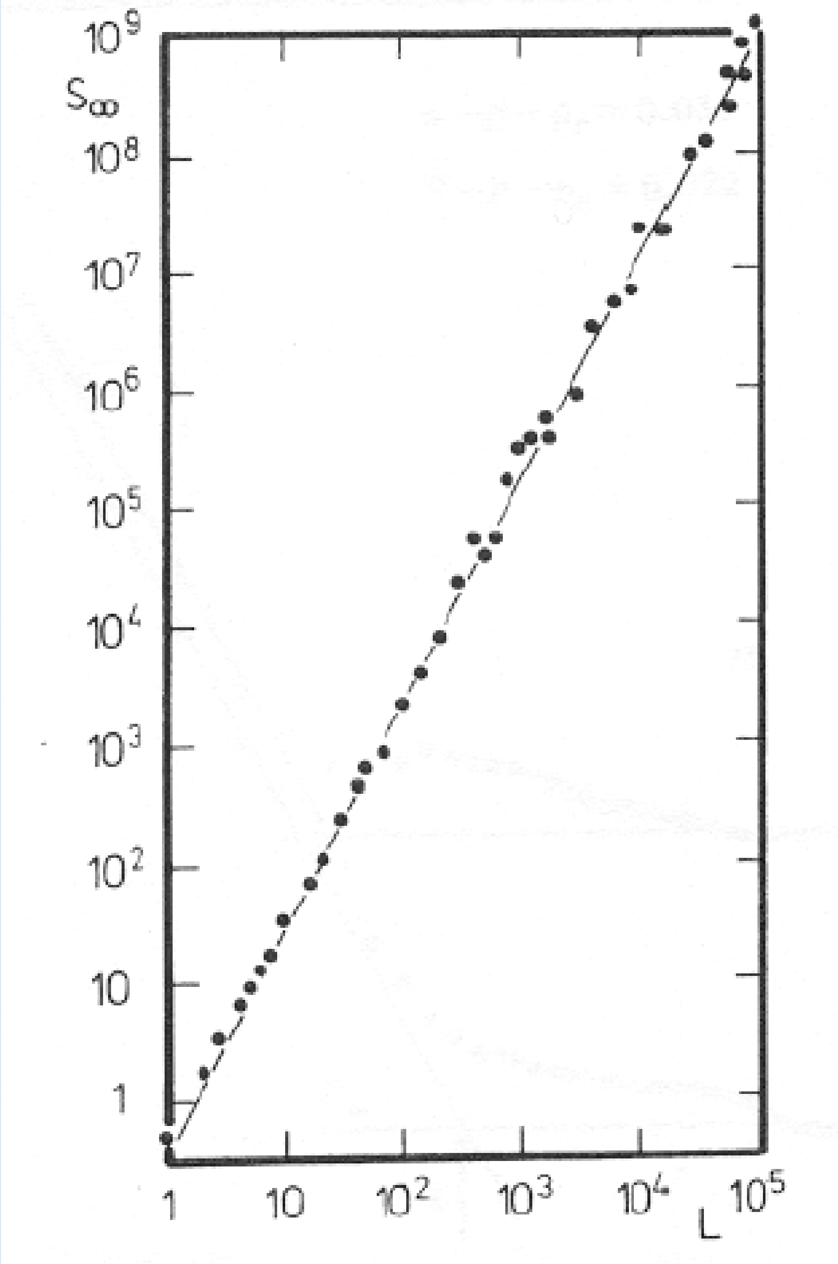
$$M(r) = Ar^{d_f}, \quad \log M(r) = \log A + d_f \log r$$

This method is analogous to the determination of d_f in deterministic fractals.

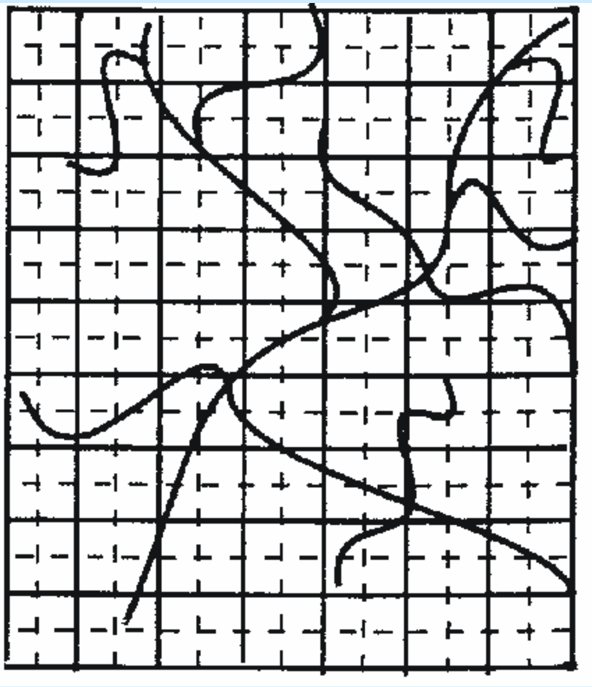
How the mass M scales with the linear metric r.

Fractal Dimension- Infinite Percolation Cluster

Slope=1.896=d_f



Box counting method



- Draw a lattice of squares of different sizes ε
- For each ε count the number of boxes $N(\varepsilon)$ needed to cover the fractal
- $N(\varepsilon)$ increases with decreasing

The fractal dimension is obtained from

$$N(\varepsilon) = A\varepsilon^{-d_f}$$

$$\log N(\varepsilon) = \log A - d_f \log \varepsilon$$

- Plotting $N(\varepsilon)$ vs ε on log-log graph – the slope is $-d_f$

Correlation method

Measurements of the density-density autocorrelation function

$$C(\mathbf{r}) = \langle \rho(\mathbf{r}')\rho(\mathbf{r}'+\mathbf{r}) \rangle_{\mathbf{r}'} = \frac{1}{V}\sum_{\mathbf{r}'} \rho(\mathbf{r}')\rho(\mathbf{r}'+\mathbf{r})$$

$$\rho(\mathbf{r}') = \begin{cases} 1 & \textit{if at } \mathbf{r}' \textit{ there is a site of the fractal} \\ 0 & \textit{if at } \mathbf{r}' \textit{ there is no site} \end{cases}$$

The volume $V = \sum_{\mathbf{r}'} \rho(\mathbf{r}')$.

$C(\mathbf{r})$ is the average density at distance $\mathbf{r}$ from a site on a fractal.

For isotropic fractals we expect $C(\mathbf{r}) = C(r) = Ar^{-\alpha}$.

The mass within a radius $\mathbf{r}$ is:

$$M(R) = \int_0^R C(r)d^d r = R^{-\alpha+d} \equiv R^{d_f}$$

$$\Rightarrow \boxed{\alpha = d - d_f}$$

Thus, from measuring α one can determine d_f

4.4 Experimental method

- Scattering experiments like x-rays, neutron scattering etc. with different wave vectors is proportional to the structure factor.

  \- The structure factor is the Fourier transform of the density-density correlation function.

For fractals – the structure factor is

$$S(\mathbf{q}) = S(q) = q^{-d_f}$$

$q = \frac{4\pi}{\lambda}\sin\vartheta$ is the wave vector.

Since physical fractals have lower and upper bounds length scales (λ_- and λ_+)

It follows that only for $\frac{4\pi}{\lambda_+}\sin\alpha < q < \frac{4\pi}{\lambda_-}\sin\vartheta$, we obtain d_f

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Measurements of $S(q)$ yields d_f

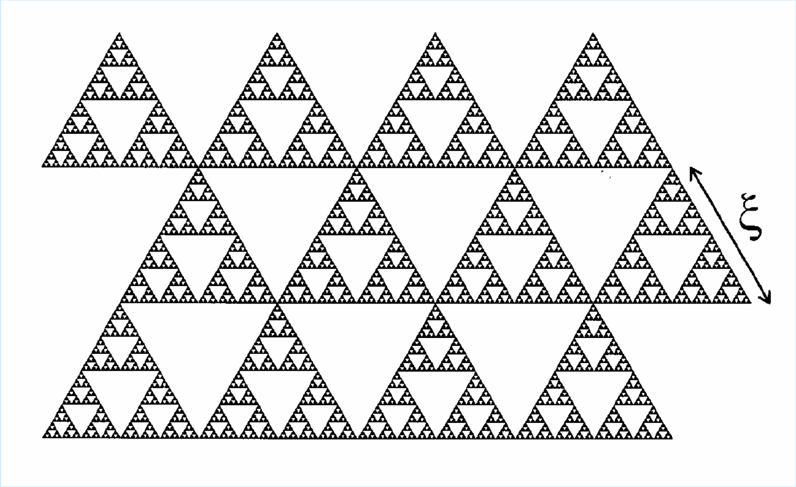
Example: polymers.

Percolation – fractal dimension

- The fractal dimension d_f describes how the mass M(r) scales within a circle of radius r

$$M(r) \sim Ar^{d_f}$$

- The center of the circle on a site
- M(r) is averaged of many different circles
- Size of finite clusters (≡holes) is ξ - correlation length
- At $p \rightarrow p_c,\ \xi \rightarrow \infty$, and we have holes of all scales
- Above p_c, ξ is finite and self-similarity exists only for scales smaller than ξ
- Above ξ - the cluster is homogeneous!



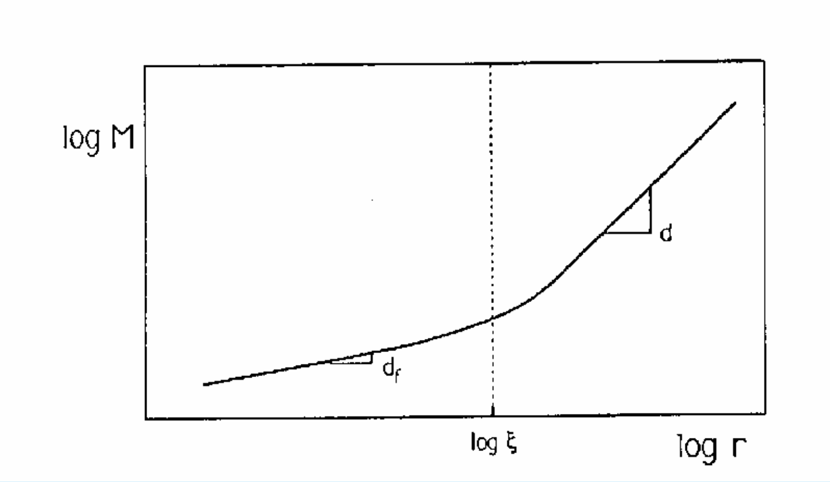
- Demonstration of self-similarity for scales below ξ and homogeneous above ξ

Percolation – Fractal Dimension

- Mathematically:

$$M(r) \sim \begin{cases} r^{d_f} & r \ll \xi \\ r^{d} & r \gg \xi \end{cases}$$

Fractal dimension - Theory



- Relation between d_f and β and ν:

We can calculate:

$$P_\infty \sim \frac{r^{d_f}}{r^d} \quad for \quad r \le \xi$$

- The probability that a site belongs to ∞-cluster is the ratio between the number of sites on the ∞-cluster (r^{d_f}) and the total number of sites (r^d)

$$\Rightarrow \quad P_\infty \approx \frac{\xi^{d_f}}{\xi^d} \Rightarrow (p-p_c)^\beta \approx \frac{(p-p_c)^{-\nu d_f}}{(p-p_c)^{-\nu d}}$$

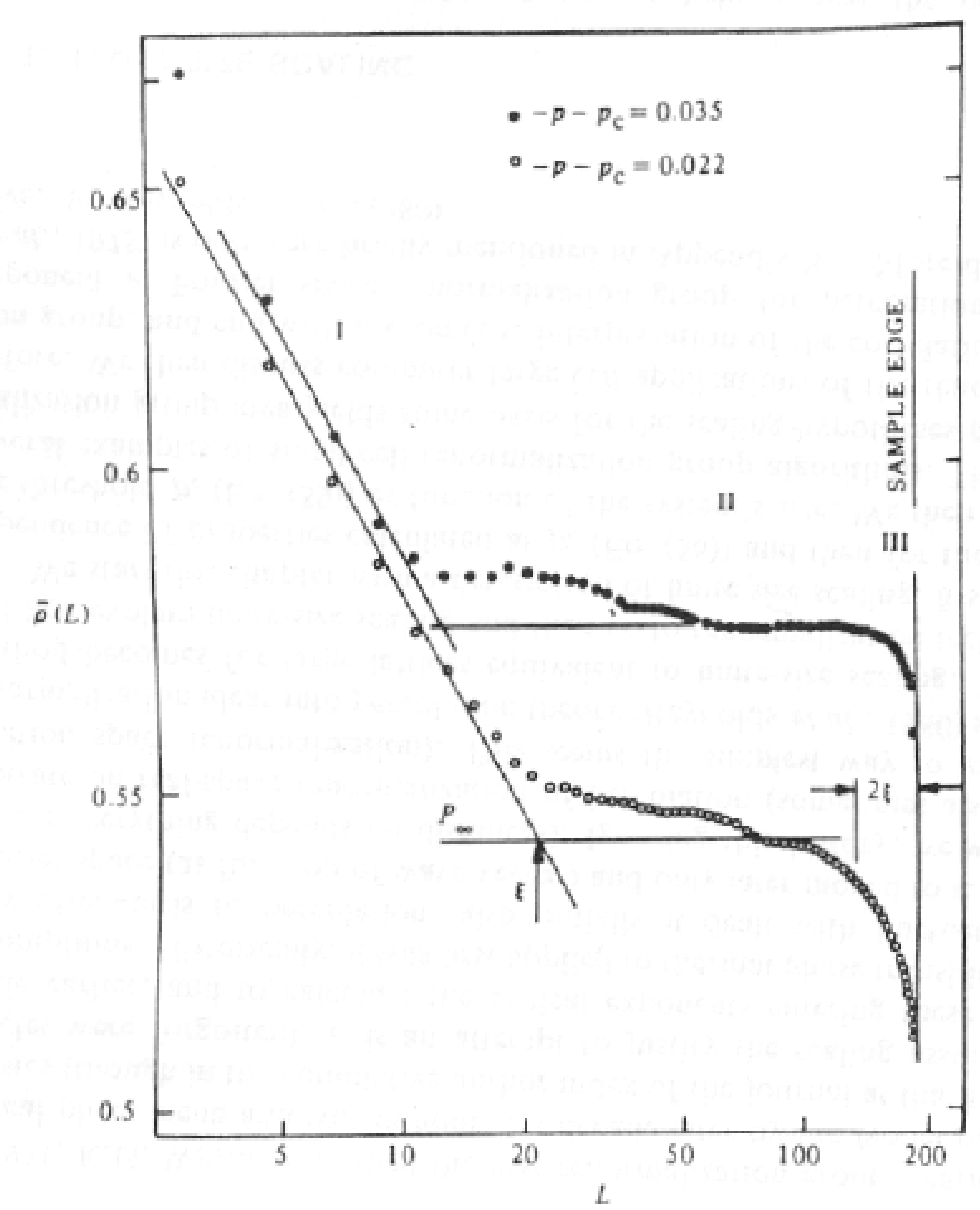
$$\Rightarrow \quad \beta = -\nu d_f + \nu d \quad \Rightarrow$$

$$\boxed{d_f = d - \frac{\beta}{\nu}}$$

Infinite Percolation Cluster

$$\rho(L) = M / L^2$$

Slope=-0.1



Fractal Dimension

$$d_f = d - \frac{\beta}{\nu}$$

$$For \quad d = 2 : \beta = 5/36, \nu = 4/3 \quad \Rightarrow \quad d_f = 2 - \frac{5 \cdot 3}{36 \cdot 4} = 2 - \frac{5}{48} = \frac{91}{48} \approx 1.896$$

$$For \quad d = 3 : \beta = 0.42, \nu = 0.88 \quad \Rightarrow \quad d_f = 3 - \frac{0.42}{0.88} \approx 2.55$$

$$For \quad d \geq 6 : \beta = 1, \nu = 1/2 \quad \Rightarrow \quad d_f = 6 - \frac{1}{1/2} = 4$$

- d_f=4 for all $d \geq 6$
- d_c=6 is the upper critical dimension
- Same d_f is for finite clusters at $p \geq p_c$ and $p < p_c$

Important relation: