Scaling Relations

$$2-\alpha=(\tau-1)/\sigma \quad \beta=(\tau-2)/\sigma \quad \gamma=\frac{3-\tau}{\sigma}$$

In the earlier 3 scaling relations ν did not appear.

We have now six exponents: α, β, γ, σ, τ and ν and another scaling relation:

For clusters with *s* sites, the rms distance between all pairs of sites on each cluster, averaged over all clusters of size *s*, is

$$R_s^2=\frac{2}{s(s-1)}\sum_{i=1}^{s}\sum_{j=1}^{i}\overline{(r_i-r_j)^2}$$

To find ξ we have to average over all cluster sizes,

$$\xi^2=\sum_{s=1}^{\infty}R_s^2 s^2 n_s \Big/ \sum_{s=1}^{\infty}s^2 n_s$$

The factor s^2 gives the same weight to each pair of sites.

Scaling Relations

Close to p_c, the large clusters dominate the sum in ξ^2.

Their mass s is related to R_s by $R_s \sim s^{1/d_f}$,

$$\xi^2 \sim \sum_{s=1}^{\infty} s^{2/d_f+2-\tau} f_\pm\left(|p-p_c|^{1/\sigma} s\right) / \sum_{s=1}^{\infty} s^{2-\tau} f_\pm\left(|p-p_c|^{1/\sigma} s\right)$$

To calculate the sums we transform them into integrals. Since 2 -τ is greater than -1, the integrations are over nonsingular integrands,

$$\xi^2 \sim |p-p_c|^{-2/(d_f\sigma)}$$

Thus we get a relation between ν, σ and τ

$$\nu = \frac{1}{d_f\sigma}$$

Scaling Relations

$$2-\alpha=(\tau-1)/\sigma \qquad \beta=(\tau-2)/\sigma \qquad \gamma=(3-\tau)/\sigma$$

$$\nu=\frac{1}{d_f\sigma}$$

Thus, we have *four* relations between the *six* exponents (α, β, γ, σ, τ and ν), and only two independent exponents.

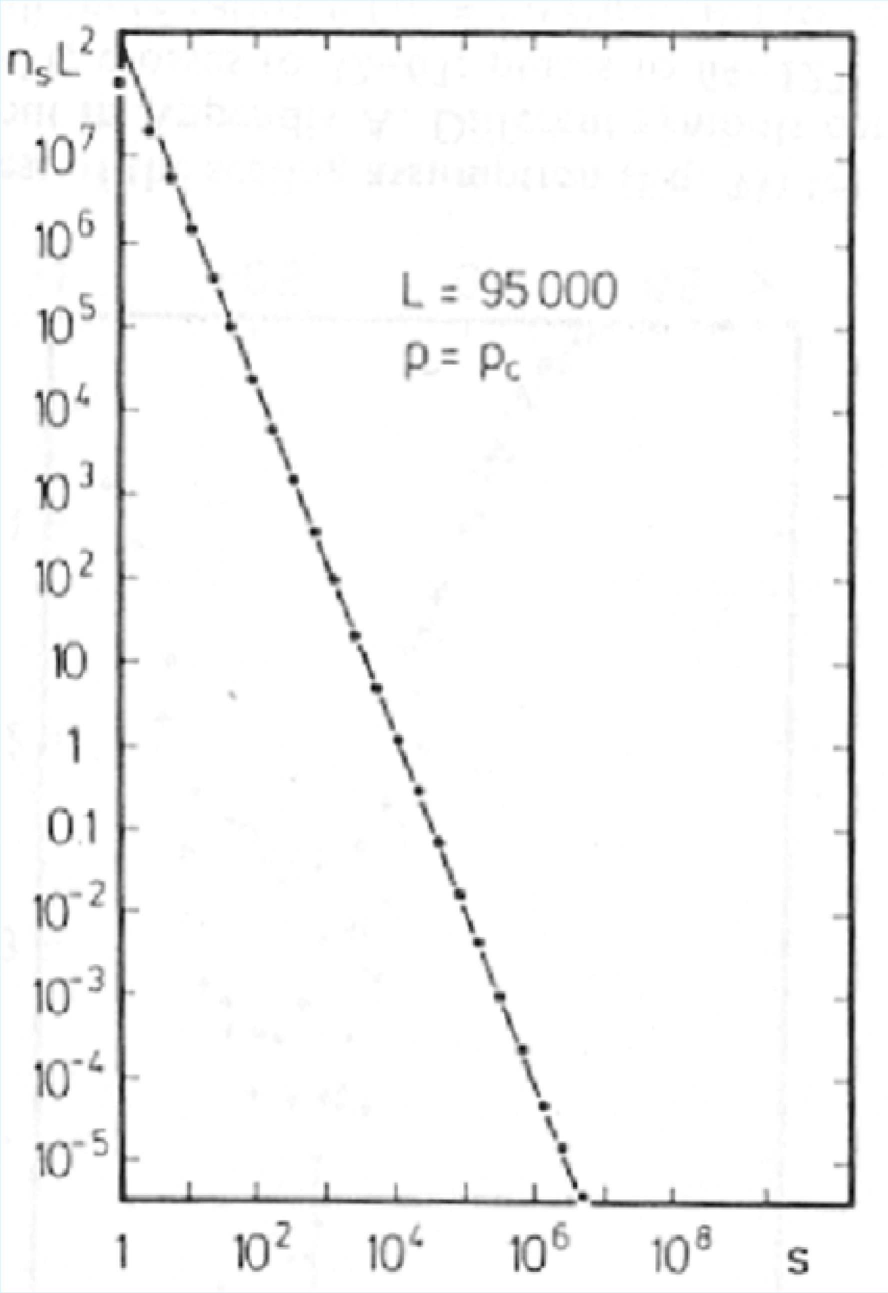
$$\rightarrow \frac{\beta}{\nu}=(\tau-2)d_f=(\tau-2)(d-\frac{\beta}{\nu})$$

From this follows:

$$\tau=1+\frac{d}{d_f}$$

$$\nu=\frac{1}{d_f\sigma}=\frac{\tau-1}{d\sigma}$$

Slope= $-\tau = -2.05$



Scaling Relations

A second quantity which characterizes the size of a finite cluster is the mean square cluster radius R_s^2, defined as

$$R^2 = \sum_{s=1}^{\infty} R_s^2 s n_s / \sum_{s=1}^{\infty} s n_s$$

Here the same weight is given to each site of the cluster, and not to each pair of sites as in ξ^2. Following the treatment of as for ξ^2 we obtain

$$R^2 \sim \left| p - p_c \right|^{-2\nu+\beta}$$

Correlation Length

We show now that the correlation length ξ is the only characteristic length scale in percolation.

The argument $z = |p - p_c|^{1/\sigma} s$ of the scaling function $f_{\pm}(z)$ can be written as $z = s / \xi^{d_f}$, and $n_s(p)$ becomes

$$n_s(p) \sim s^{-\tau} f_{\pm}(s / \xi^{d_f})$$

or equivalently, in terms of ξ,

$$n_s(p) \sim \xi^{-\tau d_f} (s / \xi^{d_f})^{-\tau} f_{\pm}(s / \xi^{d_f}) = \xi^{-d-d_f} F_{\pm}(s / \xi^{d_f})$$

where $F_{\pm}(z) = z^{-\tau} f_{\pm}(z)$. These equations show that the correlation length ξ represents the only characteristic length scale near the percolation threshold: the cluster distribution function $n_s(p)$ depends on *s* via only the ratio s / ξ^{d_f} or, on replacing *s* by $R_s^{d_f}$, on only the ratio R_s / ξ .

Scaling Theory

The sums calculated so far are special cases of the more general expression

$$M_k = \sum_{s=1}^{\infty} s^k n_s(p) \sim \sum_{s=1}^{\infty} s^{k-\tau} f_{\pm}(s/\xi^{d_f})$$

from which all relations between the exponents can be obtained.

M_k is the k moment of $n_s(k)$

The sum is transformed into the integral

$$M_k \approx \int_1^{\infty} s^{k-\tau} f_{\pm}(s/\xi^{d_f})ds \approx \xi^{d_f(k-\tau+1)} \int_{\xi^{-d_f}}^{\infty} z^{k-\tau} f_{\pm}(z)dz$$

As long as the integrand is nonsingular, $k-\tau > -1$ the lower integration limit can be extended to zero, yielding

$$M_k \sim \xi^{d_f(k-\tau+1)} \sim |p-p_c|^{(\tau-1-k)/\sigma}$$

Scaling Theory

For $k < \tau - 1$ this procedure does not work, since the lower limit dominates the integral. In this case, one can consider derivatives of M_k with respect to ξ^{-d_f}:

$$\frac{d^n M_k}{(d\xi^{-d_f})^n} \sim \xi^{d_f(k-\tau+n+1)} \int_{\xi^{-d_f}}^{\infty} z^{k-\tau+n} \frac{d^n f_{\pm}(z)}{dz^n} dz$$

where *n* is the smallest integer greater than $\tau - k - 1$. The integrand is nonsingular and hence

$$\frac{d^n M_k}{(d\xi^{-d_f})^n} \sim \xi^{d_f(k-\tau+n+1)}$$

Thus, we obtain by simple integration, up to lowest-order terms,

$$M_k \sim \xi^{d_f(k-\tau+1)} \sim |p - p_c|^{(\tau-1-k)/\sigma}$$

for all *k* values.

Scaling - Summary

$$M_0 \sim |p-p_c|^{(\tau-1)/\sigma} \sim |p-p_c|^{2-\alpha} \quad ; \quad M_1 \sim |p-p_c|^{(\tau-2)/\sigma} \sim |p-p_c|^{\beta} ;$$

$$M_2 \sim |p-p_c|^{(\tau-3)/\sigma} \sim |p-p_c|^{-\gamma}$$

From these follow $2-\alpha = (\tau-1)/\sigma$, $\beta = (\tau-2)/\sigma$, $\gamma = (3-\tau)/\sigma$.

These three relations constitute, together with $d\nu = (\tau-1)/\sigma$, four relations between the six exponents: Two independent exponents!

From these relations follow, $$d\nu = 2\beta + \gamma$$

This relation has been found useful by Toulouse (1974) to obtain the upper critical dimension d_c for percolation. At d_c, $\nu = 1/2$, $\beta = 1$, $\gamma = 1$ (solved e.g., for CT) and hence d_c=6.

The same argument leads to d_c=4 in Ising systems, where β=1/2 at critical dimension.

Scaling and Crossover Phenomena

The correlation length ξ is the only characteristic length scale in percolation. Above p_c, ξ is finite, and we expect different behavior on length scales $r < \xi$ and $r > \xi$.

We present a scaling theory for the crossover behavior in several quantities such as P_∞, $M(r)$, $M(\ell)$, $R(\ell)$. Assume the scaling ansatz

$$P_\infty \sim (p - p_c)^\beta G(r/\xi) \sim \xi^{-\beta/\nu} G(r/\xi)$$

The scaling function G describes the crossover from $r/\xi \ll 1$ to $r/\xi \gg 1$. To obtain the expected results

$$P_\infty \sim (p-p_c)^\beta \quad for \quad r \gg \xi \quad and \quad P_\infty \sim r^{d_f} / r^d \quad for \quad r \ll \xi$$

in the two limits, we must require that

$$G(x) \sim \begin{cases} x^{d_f - d}, & x \ll 1 \\ const, & x \gg 1 \end{cases}$$

Thus, we find a unique function for $P_\infty(r,\xi)$ as a function of ξ and r

Scaling and Crossover Phenomena

We also determine how the mean mass *M* of the infinite cluster scales with r and ξ above p_c.

Since $M \sim r^d P_\infty(r,\xi)$, the mean mass of the infinite cluster scales as

$$M \sim r^{d-\beta/\nu} H\left(r/\xi\right), \qquad H(x) = x^{\beta/\nu} G(x)$$

Since

$$G(x) \sim \begin{cases} x^{d_f - d}, & x << 1 \\ const, & x >> 1 \end{cases}$$

we recover $M \sim r^{d_f} \;\; for \;\; r < \xi \;\; and \;\; M \sim r^d \;\; for \;\; r > \xi.$