Scaling and Crossover Phenomena

The correlation length ξ is the only characteristic length scale in percolation. Above p_c, ξ is finite, and we expect different behavior on length scales $r < \xi$ and $r > \xi$.

We present a scaling theory for the crossover behavior in several quantities such as P_∞, $M(r)$, $M(\ell)$, $R(\ell)$. Assume the scaling ansatz

$$P_\infty \sim (p - p_c)^\beta G(r/\xi) \sim \xi^{-\beta/\nu} G(r/\xi)$$

The scaling function G describes the crossover from $r/\xi \ll 1$ to $r/\xi \gg 1$. To obtain the expected results

$$P_\infty \sim (p - p_c)^\beta \quad for \quad r \gg \xi \quad and \quad P_\infty \sim r^{d_f} / r^d \quad for \quad r \ll \xi$$

in the two limits, we must require that

$$G(x) \sim \begin{cases} x^{d_f - d}, & x \ll 1 \\ const, & x \gg 1 \end{cases}$$

Thus, we find a unique function for $P_\infty(r,\xi)$ as a function of ξ and r

Scaling and Crossover Phenomena

We also determine how the mean mass *M* of the infinite cluster scales with r and ξ above p_c.

Since $M \sim r^d P_\infty(r,\xi)$, the mean mass of the infinite cluster scales as

$$M \sim r^{d-\beta/\nu} H\left(r/\xi\right), \qquad H(x) = x^{\beta/\nu} G(x)$$

Since

$$G(x) \sim \begin{cases} x^{d_f - d}, & x << 1 \\ const, & x >> 1 \end{cases}$$

we recover $M \sim r^{d_f}$ $for\ r < \xi\ and\ M \sim r^d\ for\ r > \xi.$

Finite Size Scaling

The dependence critical properties near p_c and p_c itself on the system size L

We had the scaling for $$P_\infty \sim \xi^{-\beta/\nu} G\left(L/\xi\right) \sim \begin{cases} \xi^{-\beta/\nu} & L >> \xi \\ L^{-\beta/\nu} & L << \xi \end{cases}$$

We expect similar scaling for other properties:
In general for property Y(L,p) we expect

$$Y(L,p) \sim \xi^{-y/\nu} F_1\left(L/\xi\right) \sim \begin{cases} \xi^{-y/\nu} & L >> \xi \\ L^{-y/\nu} & L << \xi \end{cases}$$

From which follows:

$$Y(L,p) \sim (p-p_c)^y F_2((p-p_c)L^{1/\nu}) \sim L^{-y/\nu} F_3\left((p-p_c)L^{1/\nu}\right)$$

Finite Size Scaling

- If we calculate Y exactly at p=pc as a function of L we can get the exponent y/ν

•Example $$P_\infty(p_c, L) \sim L^{\beta/\nu};\; M(p_c, L) \sim L^{d-\beta/\nu}$$

- If ν is known we find y.

Example d=1: we use $S(L,p) = L^{\gamma/\nu} f_s\left((p-p_c)L^{1/\nu}\right)$

We showed that for L → ∞

The mean cluster size $$S = \frac{1+p}{1-p} \sim \xi \Rightarrow \gamma = \nu$$

For p=1 and finite L, a single cluster exist S=L

Thus $\dfrac{\gamma}{\nu} = 1$ From which follows $\gamma = \nu = 1$

p_c in Finite Systems

For L finite there is a finite probability to find a spanning cluster at any finite *p*.

E.g., in d=1: The probability to find a cluster of size L is $\phi_p(L) = p^L = e^{-L/\xi}$
Which is finite for all *p*!!

If L<ξ the prob. is greater than $1/e$

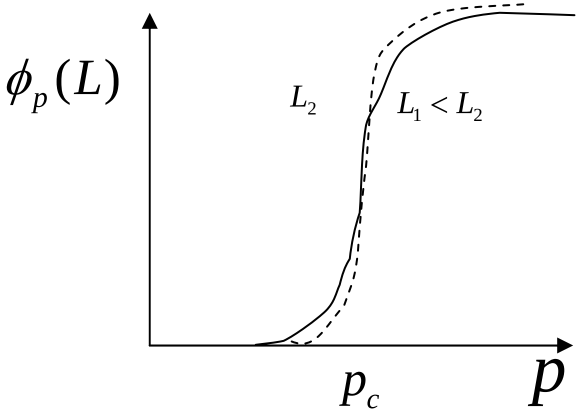
L>ξ the prob. is smaller than $1/e$

For L→∞ we expect Ø=0 below p_c and Ø=1 above p_c → Step function

In d=1 Ø=0 for p<1 and Ø=1 for p=1.

p_c in Finite Systems

For L finite use expect:



Since $\phi_p(L)$ is approaching a step function for L→∞, we define an effective threshold p^* when $\phi_{p^*}(L) = \frac{1}{2}$. When L→∞ $p^*(L) \to p_c$

For d=1 $\phi_{p^*}(L) = e^{-L/(1-p^*)} = \frac{1}{2}$

$$\Rightarrow 1 - p^* \equiv p_c - p^* \approx \frac{1}{L}$$

* How p^* will behave in d- dimensional percolation?

p_c for Finite Systems

$\phi_p(L)$: the probability for percolation in a d-dimensional lattice of size L with prob. p.

Since $\phi_p(\infty)=1$ for p>p_c and $\phi_p(\infty)=0$ for p<p_c,

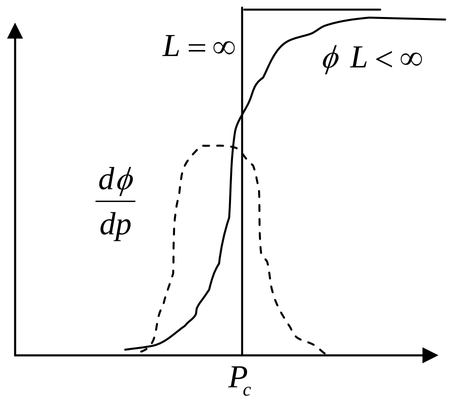
we expect for $\phi_p(L)$ y=0 (due to step function) and

$$\phi_p(L) = f\left((p-p_c)L^{1/\nu}\right) \equiv f(x)$$

f(x) increases from 0 to 1 for x increasing from -∞ to ∞

The quantity $\dfrac{d\phi}{dp}$ is the probability that percolation occurs (for the first time) between *p* and *p*+*dp* per unit *dp*.

p_c in Finite Systems



The prob. that at p percolation occur, for the first time :

(prob. that p is p_c)
$$\frac{d\phi}{dp} = L^{1/\nu} f'\left((p-p_c)L^{1/\nu}\right)$$

For $L \to \infty$
$$\frac{d\phi}{dp} \to \delta(p-p_c)$$

The average critical concentration $\overline{p}$ as a function of L is given by

$$\overline{p} = \int_0^1 p \frac{d\phi}{dp} dp$$

$\overline{P}$ can be easily calculated by simulation for different L.

From the form of $\frac{d\phi}{dp}$ and since $\int_0^1 \frac{d\phi}{dp} dp = 1$ it follows

$$\overline{p} - p_c = C \cdot L^{-1/\nu} \qquad C \equiv \int_{-\infty}^{\infty} z f'(z) dz \qquad z \equiv (p-p_c)L^{1/\nu}$$

p_c for Finite Systems

$$\overline{p} - p_c = CL^{-1/\nu} \qquad (1)$$

In some symmetrical cases such as triangular lattice $f'(z)$ is symmetric around z=0 and thus $C \equiv \int_{-\infty}^{\infty} z f'(z) dz = 0$

and there is no correction! $\overline{p} = p_c$

Eq. (1) helps to determine in simulations both p_c and ν

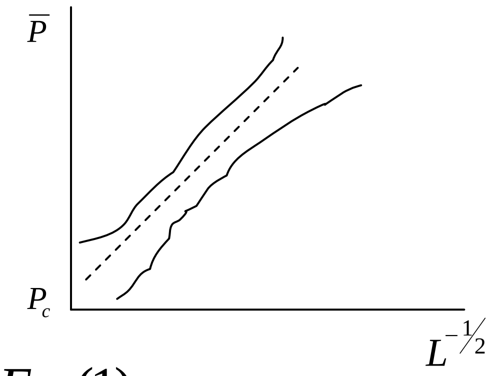
Plot the measured $\overline{p}$ as a function of $L^{-1/\nu}$

Trying several values of ν when $\nu' = \nu$ one gets a straight line which also determine p_c

Example d=1: $$\frac{d\phi}{dL} = Lp^{L-1}$$

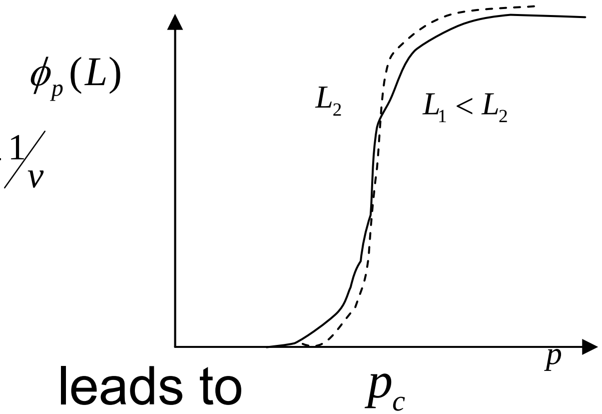
$$\overline{p} = \frac{L}{L+1}$$

$$p_c - \overline{p} = 1 - \overline{p} = \frac{1}{L+1} \qquad as \quad in \quad Eq.(1)$$



p_c for Finite Systems

$$\overline{p} - p_c = CL^{-1/\nu} \qquad (1)$$



• Every $\phi_{p^*}(L) = cons.$ leads to $p^* - p_c = c_1^* L^{-1/\nu}$

•Also $\left.\dfrac{d\phi}{dp}\right|_{p^*} = \max$ (p^* is an inflection point) leads to

$$p^* - p_c = c_2^* L^{-1/\nu}$$

• The maximum of the mean finite clusters size S(p) approach to p_c as $L^{-1/\nu}$

• Eq.(1) is valid for every percolation property.

• The above conclusions are correct also for other critical phenomena used earlier.

p-p_c is replaced by T-T_c

The width of transition in finite systems

Define width of transition Δ(L,p)

$$\Delta^2 = \int (p-\overline{p})^2 \frac{d\phi}{dp} dp = \overline{p^2} - \overline{p}^2$$

$$\propto \; L^{-2/\nu}$$

$$\Delta \sim \; L^{-1/\nu} \qquad\qquad (2)$$

Δ can be determined from MC simulations and thus evaluate ν

Here the knowledge of p_c is not needed!

To determine p_c one can calculate

$$\overline{p} - p_c \;\propto\; \Delta$$

Eq. (2) will be useful for optimization problems!!