Box dimension

Different ways to define fractal dimensions usually lead to the same result.

Important: different ways usually lead to different methods to calculate the fractal dimension, in particular, in random fractals.

We define the box dimension:

- Given a set of points in d-dimensions.
- Calculate the number of boxes of linear size ε needed to cover the set.
- If $N(\varepsilon)$ is the number of boxes of size ε and there exists the relation

$$N(\varepsilon) = \frac{A}{\varepsilon^{d_f}} \quad (\textit{for } \varepsilon \to 0)$$

- Then $d_f = \dfrac{\log N(\varepsilon)}{\log \dfrac{1}{\varepsilon}} \quad (\textit{for } \varepsilon \to 0)$

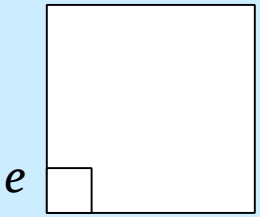
  is the fractal dimension of the set.

Box dimension

For a line section: $N(e)=\frac{A}{e}$

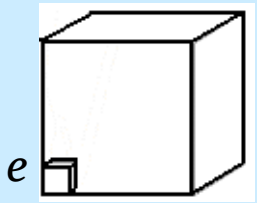
$\Rightarrow d_f = 1$

For a square: $N(e)=\frac{A}{e^2}$



$\Rightarrow d_f = 2$

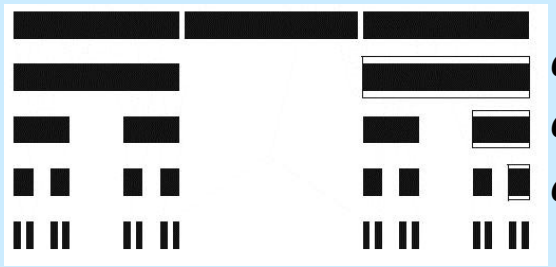
For a cube: $N(e)=\frac{A}{e^3}$



$\Rightarrow d_f = 3$

That is for integer dimensions $d_f = d$ as expected !

Triadic Cantor set



$e = 1/3$

$e = (1/3)^2$

$e = (1/3)^3$

For boxes of size $\left(\frac{1}{e}\right)^k$

The number is: $N(e) = 2^k$

Solving $2^k = \frac{A}{e^{d_f}} \Rightarrow d_f = \frac{\ln N(e)}{\ln \frac{1}{e}} = \frac{\ln 2^k}{\ln 3^k} = \frac{\ln 2}{\ln 3} \cong 0.6309$

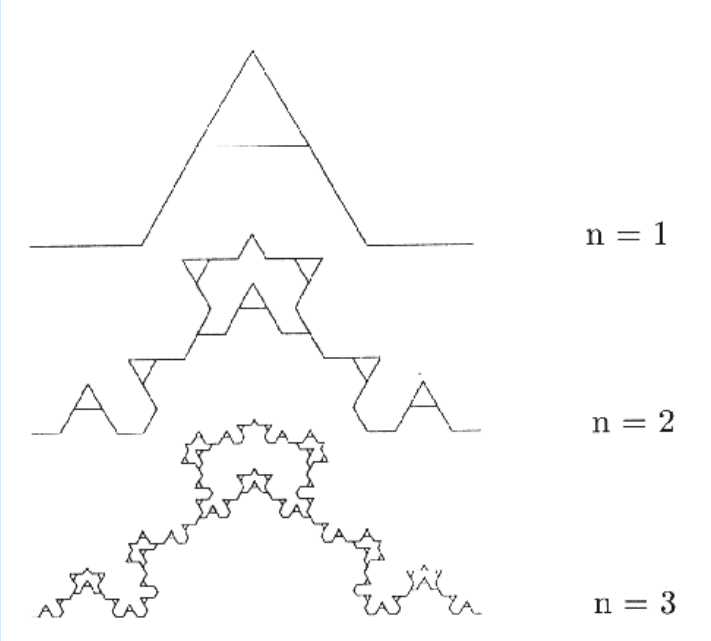
More fractal dimensions

- The common fractal dimension d_f cannot fully characterize the fractal
- Given fractal $\Rightarrow \not\Leftarrow$ fractal dimension
- More fractal dimensions are needed!
- How many dimensions are needed – no answer today

Shortest path (chemical distance) dimension - $d_{\min}$

- The fractal dimension of the shortest path defined by $l(bL) = b^{d_{\min}} l(L)$

Example: modified Koch curve



$$d_f : \; M\left(\frac{1}{4}L\right) = \frac{1}{7}M(L) = \left(\frac{1}{4}\right)^{d_f} M(L)$$

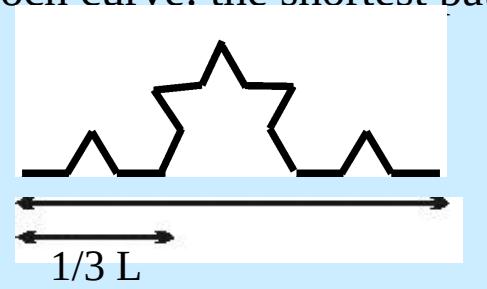
$$d_f = \frac{\log 7}{\log 4} \cong 1.404, \quad M(L) = AL^{d_f}$$

$$d_{\min} : \; l\left(\frac{1}{4}L\right) = \frac{1}{5}l(L) = \left(\frac{1}{4}\right)^{d_{\min}} l(L)$$

$$d_{\min} = \frac{\log 5}{\log 4} \cong 1.161, \quad l = BL^{d_{\min}}$$

Shortest path dimension - $d_{\min}$

For Koch curve: the shortest path is the line itself



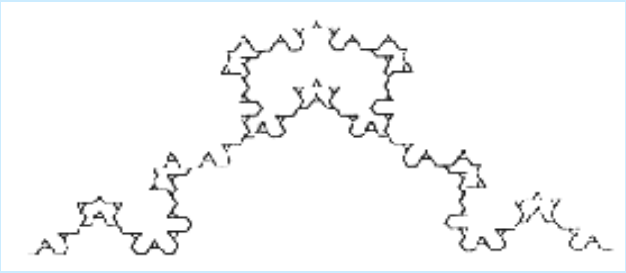
$$l\left(\frac{1}{3}L\right)=\frac{1}{4}l(L)=\left(\frac{1}{3}\right)^{d_f}l(L)$$

$$d_f=\frac{\log 4}{\log 3}$$

Chemical dimension - d_l - "how the mass scales with the shortest path"

Defined by: $M(bl)=b^{d_l}M(l),\qquad M(l)=Cl^{d_l}$

For the modified Koch curve



$$M\left(\frac{1}{5}l\right)=\frac{1}{7}M(l)=\left(\frac{1}{5}\right)^{d_l}M(l)$$

$$d_l=\frac{\log 7}{\log 5}\cong 1.209,\quad M(l)=Cl^{d_l}$$

Is there a relation between $d_l, d_{\min}$ and d_f ?

$M(L)=AL^{d_f}$ *from* $l=BL^{d_{\min}}$ *follows* $L\sim l^{1/d_{\min}}$

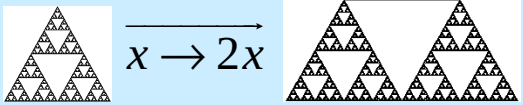
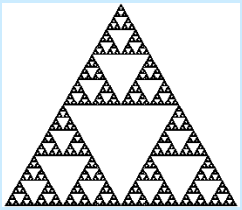
$$=A'l^{d_f/d_{\min}}$$

$$=Cl^{d_l}\Rightarrow d_l=d_f/d_{\min}$$

More characteristics of fractals include: backbone, external perimeter, red bonds, etc.

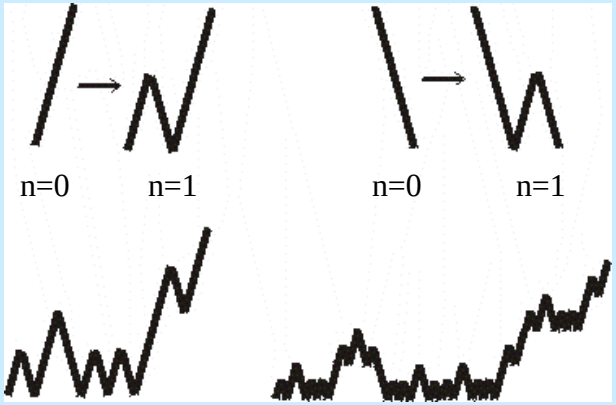
3. Self-affinity

Self-similarity or scale invariance is an isotropic property, the change of scale is the same in every direction in space.

Example: Sierpinski gasket  $\xrightarrow{x \to 2x,\ y \to 2y}$ 

Self-affinity – include anisotropic symmetry magnifying x in different scale than y.

Example:



- Here we see that to get the same picture we need to magnify the x axis by 4 and y axis by 2, $x \to 4x,\ y \to 2y$

$$\boxed{M\left(\frac{1}{4}L_x, \frac{1}{2}L_y\right) = \frac{1}{4}M(L_x, L_y)}$$

Generalization of self-similar fractals: $M(bL) = b^{d_f} M(L)$

3.1 Fractal dimension – self-affine structures

Here we need to define two fractal dimensions

$$M(aL_x, bL_y) = a^{d_f^x} M(L_x, L_y)$$

$$= b^{d_f^y} M(L_x, L_y)$$

Example:

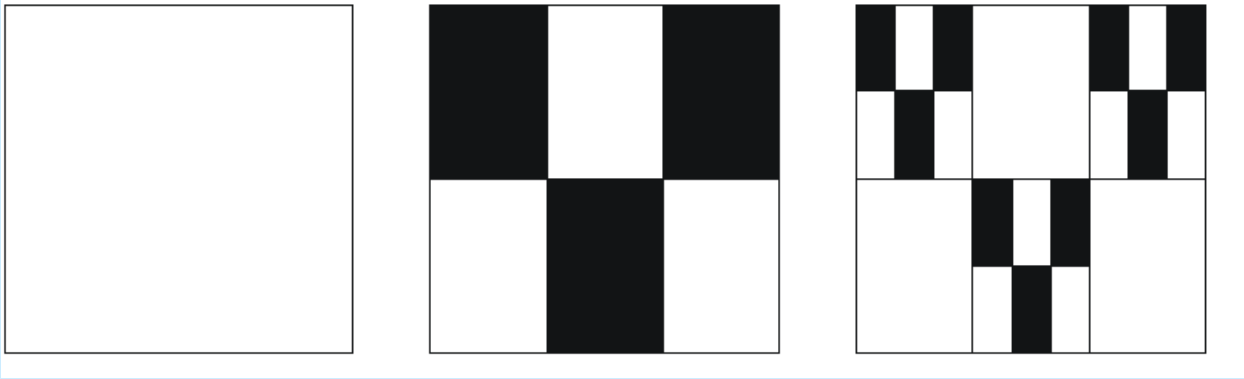
$$M\left(\frac{1}{4}L_x, \frac{1}{2}L_y\right) = \frac{1}{4}M(L_x, L_y) = \left(\frac{1}{4}\right)^{d_f^x} M(L_x, L_y)$$

$$M\left(\frac{1}{4}L_x, \frac{1}{2}L_y\right) = \frac{1}{4}M(L_x, L_y) = \left(\frac{1}{2}\right)^{d_f^y} M(L_x, L_y)$$

$$\left(\frac{1}{4}\right)^{d_f^x} = \frac{1}{4} \Rightarrow d_f^x = 1,$$

$$\left(\frac{1}{2}\right)^{d_f^y} = \frac{1}{4} \Rightarrow d_f^y = 2$$

Example: self-affine Sierpinski carpet

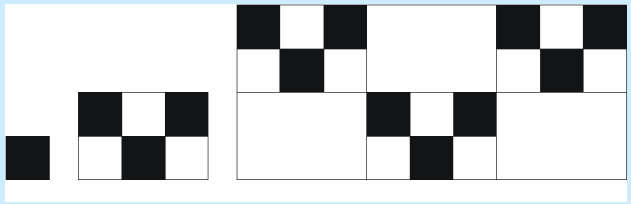


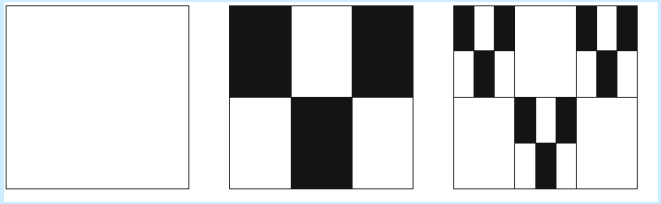
$$M\left(\frac{1}{3}L_x,\frac{1}{2}L_y\right)=\frac{1}{3}M(L_x,L_y)=\left(\frac{1}{3}\right)^{d_f^x}M(L_x,L_y)$$

$$=\left(\frac{1}{2}\right)^{d_f^y}M(L_x,L_y)$$

$$d_f^x=1,\quad d_f^y=\frac{\log 3}{\log 2}$$

Self affine fractals





Here also

$$M\left(\frac{1}{3}L_x,\frac{1}{2}L_y\right)=\frac{1}{3}M(L_x,L_y)=\left(\frac{1}{3}\right)^{d_f^x}M(L_x,L_y)$$

$$=\left(\frac{1}{2}\right)^{d_f^y}M(L_x,L_y),\qquad d_f^x=1,\quad d_f^y=\frac{\log 3}{\log 2}$$

Generalization:

Start with a square of unit size:

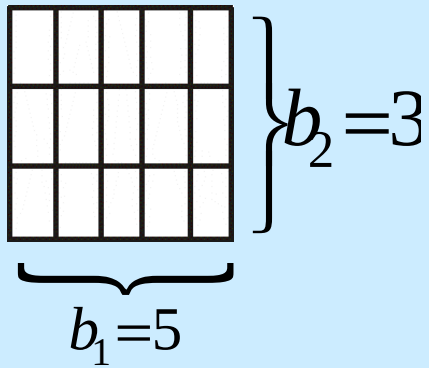
(a) Divide x axis to b_1 and y axis to b_2

(b) We get rectangulars of size $(1/b_1)\times(1/b_2)$

(c) Number of rectangulars $b_1\times b_2$

(d) Keep n rectangulars and remove $b_1\times b_2-n$ of them (above: n=3, b_2 =2, b_1 =3)

(e) To each rectangular left full, apply the same rule.



The fractal dimension:

$$M\left(\frac{1}{b_1}L_x,\frac{1}{b_2}L_y\right)=\frac{1}{n}M(L_x,L_y)=\left(\frac{1}{b_1}\right)^{d_f^x}M(L_x,L_y)$$

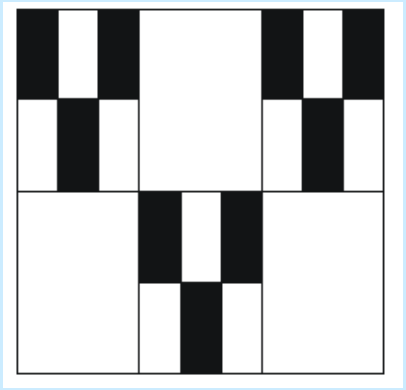
$$=\left(\frac{1}{b_2}\right)^{d_f^y}M(L_x,L_y),\qquad d_f^x=\frac{\log b_1}{\log n},\quad d_f^y=\frac{\log b_2}{\log n}$$

3.2 Local dimension – box dimension

Alternative definition of dimension is self-affine using box dimension

Example:

- Chose a square box of linear size $\frac{1}{3}, \frac{1}{3^2}, \ldots \frac{1}{3^n}$
- How many boxes are needed to cover the fractal?



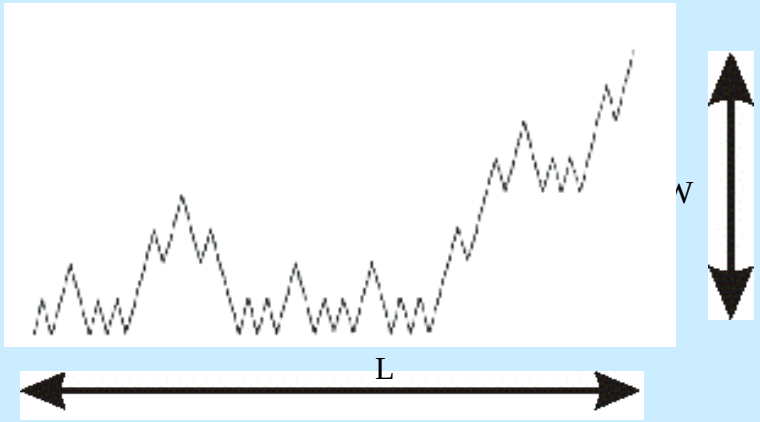
For size $\frac{1}{3}$ we need $\frac{3 \cdot \frac{1}{3} \cdot \frac{1}{2}}{\left(\frac{1}{3}\right)^2} = \frac{\text{rectangular area} \times \text{number of rectangulars}}{\text{box area}}$

In general for $\frac{1}{3^k}$ we need $N(e) = \frac{3^k \cdot \frac{1}{3^k} \cdot \frac{1}{2^k}}{\left(\frac{1}{3^k}\right)^2}$ boxes. More general: if we divide to $b_1 \times b_2$ rectangulars and leave n of them full, we obtain a box of size $e = b_1^{-k}$ and the number of boxes $N(e) = n^k \frac{b_1^{-k} \cdot b_2^{-k}}{\left(b_1^{-k}\right)^2}$

The local box dimension: $N(e) = e^{-d_f^l}$, $d_f^l = \frac{\ln N(e)}{\ln \frac{1}{e}} = \frac{k \ln \frac{nb_1}{b_2}}{k \ln b_1} = \frac{\ln \frac{nb_1}{b_1}}{\ln b_1}$

Self affine curves – single valued

Example:



Alternative definition of dimension:
Denote L – linear scale in x-direction
Denote W – linear scale in y-direction

Dimension a defined by $\boxed{W(bL) = b^a W(L)}$

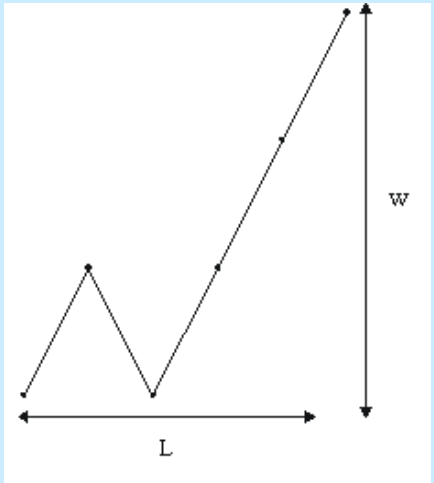
The dimension a is also called roughness exponent

For the above fractal

$$W\left(\frac{1}{4}L\right) = \frac{1}{2}W(L) = \left(\frac{1}{4}\right)^a W(L)$$

$$\Rightarrow a = \frac{\log 2}{\log 4} = \frac{1}{2}$$

For the fractal



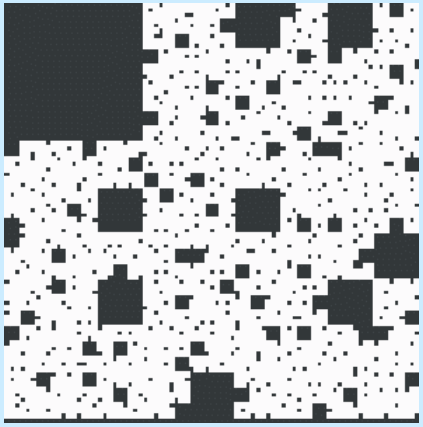
$$W\left(\frac{1}{5}L\right) = \frac{1}{3}W(L) = \left(\frac{1}{5}\right)^a W(L)$$

$$a = \frac{\log 3}{\log 5} \cong 0.683$$

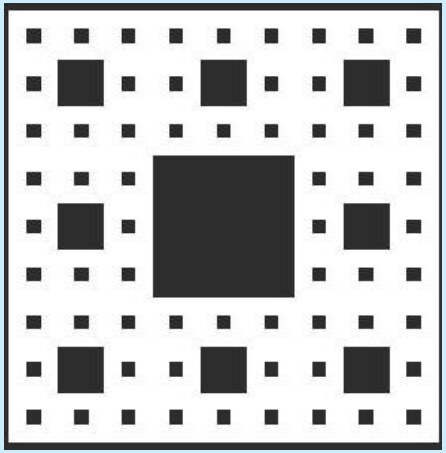
Random Fractals

- Fractals do not have to be deterministic
- One can generate random fractals
- Instead of always removing the central square, we remove randomly one of the 9 squares

Random Sierpinski carpet



Deterministic Sierpinski carpet

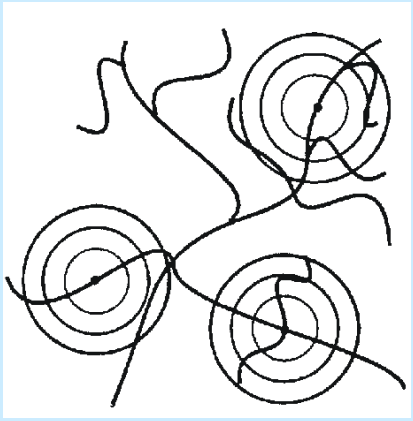


- The fractal dimension of the random Sierpinski carpet is the same as the deterministic: $M\left(\frac{1}{3}L\right)=\frac{1}{8}M(L)=\left(\frac{1}{3}\right)^{d_f}M(L), \quad d_f=\frac{\log 8}{\log 3}\cong 1.893$
- The self-similarity is not exact – valid statistically

Random Fractals – Fractal Dimension

Methods: (a) sand box; (b) box counting; (c) correlations.

4.1 Sand Box method



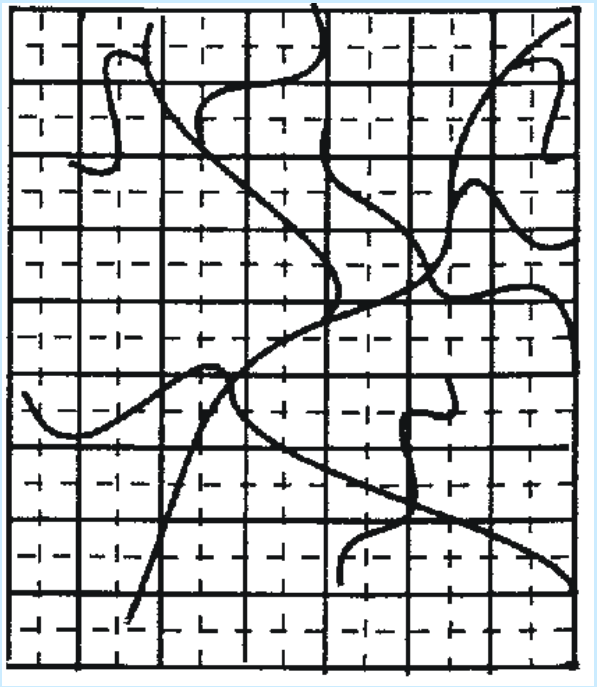
- Choose a site on the fractal – origin
- plot circles of several radiuses $r \ll R_{\max}$
- $R_{\max}$ ~ radius of the fractal
- count the number of sites inside r
- repeat the measurements for several origins
- average over all results for each r - $M(r)$
- plot $M(r)$ vs r on log-log plot
- the slope is d_f of the fractal

$$M(r) = Ar^{d_f}, \quad \log M(r) = \log A + d_f \log r$$

This method is analogous to the determination of d_f in deterministic fractals.

How the mass M scales with the linear metric r .

4.2 Box counting method



- Draw a lattice of squares of different sizes e
- For each e count the number of boxes $N(e)$ needed to cover the fractal
- $N(e)$ increases with decreasing e

The fractal dimension is obtained from

$$N(e) = Ae^{-d_f}$$

$$\log N(e) = \log A - d_f \log e$$

- Plotting $N(e)$ vs e on log-log graph – the slope is $-d_f$

4.3 Correlation method

Measurements of the density-density autocorrelation function

$$C(\mathbf{r}) = \left\langle r(\mathbf{r}')\, r(\mathbf{r}'+\mathbf{r}) \right\rangle_{\mathbf{r}'} = \frac{1}{V}\sum_{\mathbf{r}'} r(\mathbf{r}')\, r(\mathbf{r}'+\mathbf{r})$$

$$r(\mathbf{r}') = \begin{cases} 1 & \textit{if at } \mathbf{r}' \textit{ there is a site of the fractal} \\ 0 & \textit{if at } \mathbf{r}' \textit{ there is no site} \end{cases}$$

The volume $V = \sum_{\mathbf{r}'} r(\mathbf{r}')$.

$C(\mathbf{r})$ is the average density at distance $\mathbf{r}$ from a site on a fractal.

For isotropic fractals we expect $C(\mathbf{r}) = C(r) = Ar^{-a}$.
The mass within a radius R is:

$$M(R) = \int_0^R C(r) d^d r = R^{-a+d} \equiv R^{d_f}$$

$$\Rightarrow \boxed{a = d - d_f}$$

Thus, from measuring a one can determine d_f.

4.4 Experimental method

- Scattering experiments like x-rays, neutron scattering etc. with different wave vectors is proportional to the structure factor.
- The structure factor is the Fourier transform of the density-density correlation function.

For fractals – the structure factor is

$$S(\mathbf{q}) = S(q) = q^{-d_f}$$

$q = \frac{4p}{l}\sin J$ is the wave vector.

Since physical fractals have lower and upper bounds length scales (l_- and l_+)

It follows that only for $\frac{4p}{l_+}\sin a < q < \frac{4p}{l_-}\sin J$, we obtain d_f

- Measurements of $S(q)$ yields d_f

Example: polymers.