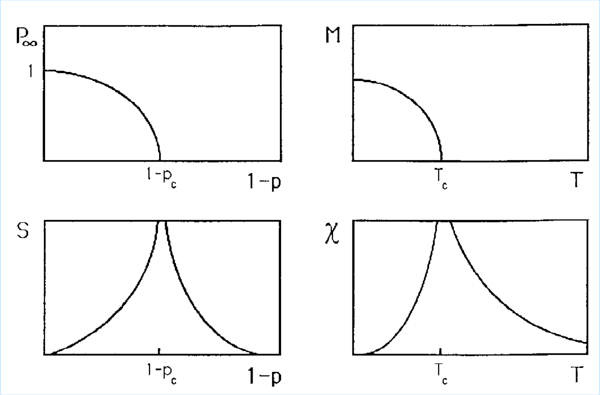
Percolation – Phase Transition



- Example of a geometrical phase transition
- p_c – critical threshold separates two phases:
  (1) ordered $p>p_c$ – infinite cluster
  (2) disordered $p<p_c$ – finite clusters
- Analogy to { thermodynamic phase transition / magnetic phase transition }

Ferromagnetic – paramagnetic phase transition

$T<T_c$ spontaneous magnetization $M>0$ – ferromagnetic phase
integration between spins $\Rightarrow$ order

$T>T_c$ no magnetization $M=0$ – paramagnetic phase
termal energy $\Rightarrow$ disorder

M – called “order parameter” scales as $M \sim (T_c - T)^b$

c - magnetic fluctuations – susceptibility

$$c \sim \left\langle (M - \bar{M})^2 \right\rangle^{1/2} \sim |T - T_c|^{-g}$$

x - correlation length (size of ordered clusters)

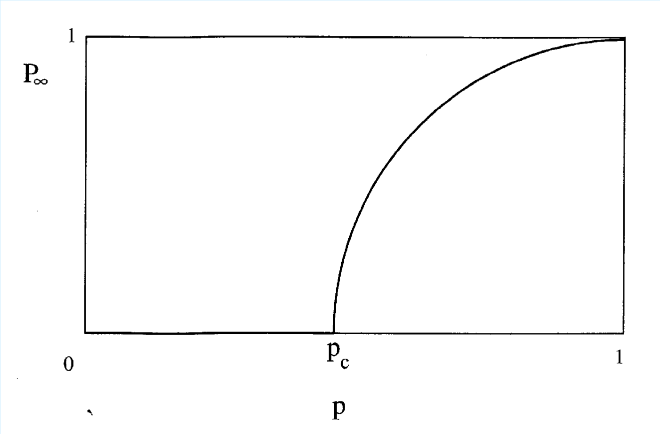
$$x \sim |T_c - T|^{-n}$$

b, g, n - called critical exponents

Percolation – critical exponent

✓p – same role as T in thermal phase transitions

✓p_∞ - probability that a site (bond) belongs to ∞ cluster
order parameter $p_\infty \propto (p - p_c)^b$ - similar to magnetization



✓x - correlation length – mean distance between two sites on the same cluster

$$x \propto | p - p_c |^{-n}$$

✓The average size of finite clusters $S \sim |p - p_c|^{-g}$
(analogous to susceptibility)

✓ b and g are the same for $p>p_c$ and $p<p_c$

✓For x and S take into account all finite clusters

✓ b, n and g called critical exponents ⇒ describe critical behavior near the transition

✓The exponents are universal

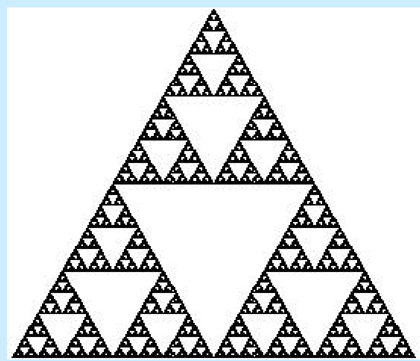
✓Universality – property of second order phase transition (order parameter →0 continuously)
All magnets in d=3 have same b
independent on the lattice and type of interactions

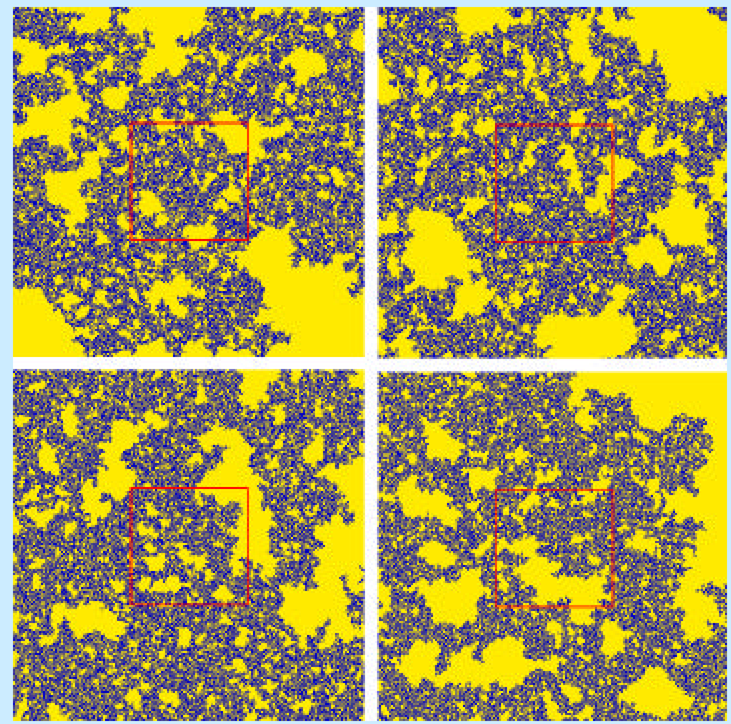
✓T_c – depends on details (interactions, lattice) – same for p_c

Percolation	$d=2$	$d=3$	$d \geq 6$
Order parameter $P_{\infty} : b$	$5/36$	0.417 ± 0.003	1
Correlation length $x : v$	$4/3$	0.875 ± 0.008	$1/2$
Mean cluster size $S : g$	$43/18$	1.795 ± 0.005	1
Magnetism	$d=2$	$d=3$	$d \geq 6$
Order parameter $m : b$	$1/8$	0.32	$1/2$
Correlation length $x : v$	1	0.63	$1/2$
Susceptibility $X : g$	$7/4$	1.24	1

Percolation – Geometrical Properties

- A percolation cluster can be characterized by fractal geometry
- We can see in the infinite cluster, at p_c, holes in all scales – like Sierpinski gasket
- The cluster is self-similar (from pixel size to system size)





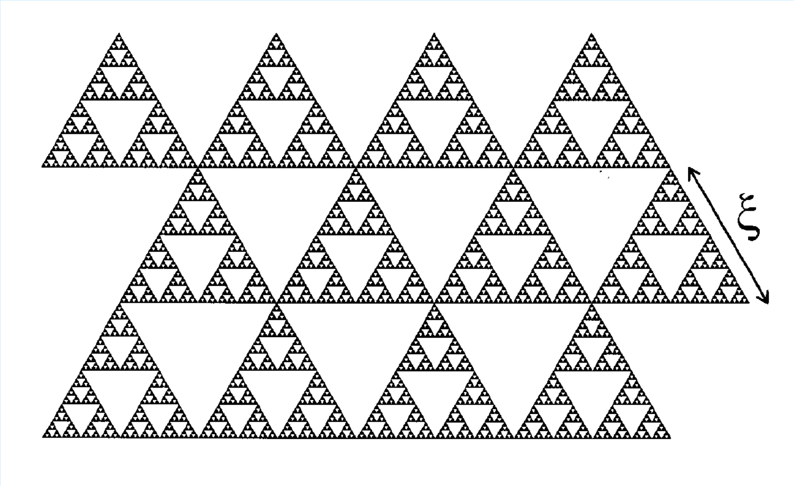
- The square in left top is magnified in right top
  $\Rightarrow$ magnified in left bottom
  $\Rightarrow$ magnified in right bottom
- The difficulty to easily realize the order is a sign of self-similarity

Percolation – fractal dimension

- The fractal dimension d_f describes how the mass M(r) scales within a circle of radius r

$$M(r) \sim Ar^{d_f}$$

- The center of the circle on a site
- M(r) is averaged of many different circles
- Size of finite clusters (≡holes) is ξ - correlation length
- At $p \to p_c,\ x \to \infty$, and we have holes of all scales
- Above p_c, ξ is finite and self-similarity exists only for scales smaller than ξ
- Above ξ - the cluster is homogeneous!

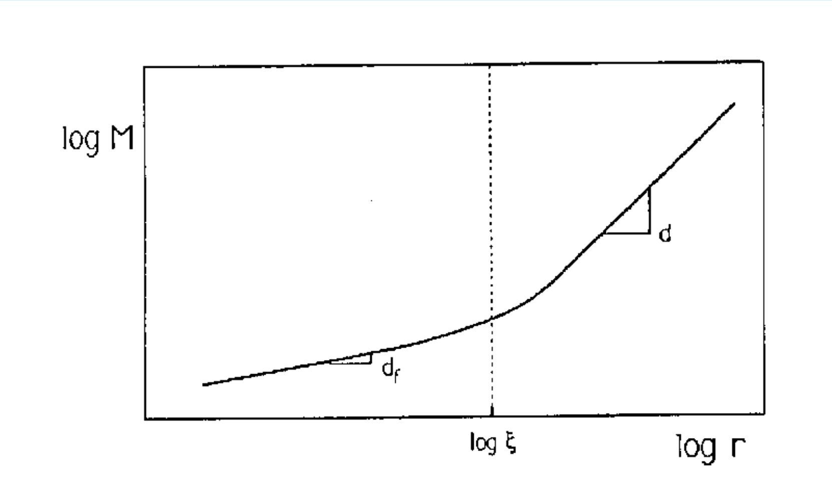


- Demonstration of self-similarity for scales below ξ and homogeneous above ξ

Percolation – Fractal Dimension

- Mathematically:

$$M(r) \sim \begin{cases} r^{d_f} & r \ll x \\ r^{d} & r \gg x \end{cases}$$



Fractal dimension - Theory

- Relation between d_f and β and ν:
  We can calculate:

$$P_\infty \sim \frac{r^{d_f}}{r^{d}} \quad for \quad r \le x$$

- The probability that a site belongs to ∞-cluster is the ratio between the number of sites on the ∞-cluster (r^{d_f}) and the total number of sites (r^{d})

$$\Rightarrow \quad P_\infty \approx \frac{x^{d_f}}{x^{d}} \Rightarrow (p-p_c)^{b} \approx \frac{(p-p_c)^{-n d_f}}{(p-p_c)^{-n d}}$$

$$\Rightarrow \quad b = -n d_f + n d \quad \Rightarrow$$

$$\boxed{d_f = d - \frac{b}{n}}$$

Fractal Dimension

$$d_f = d - \frac{b}{n}$$

$$For \quad d = 2: b = 5/36, n = 4/3 \quad \Rightarrow \quad d_f = 2 - \frac{5 \cdot 3}{36 \cdot 4} = 2 - \frac{5}{48} = \frac{91}{48} \simeq 1.896$$

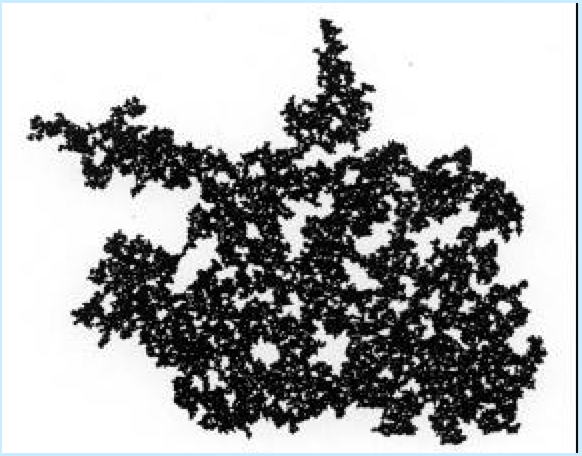
$$For \quad d = 3: b = 0.42, n = 0.88 \quad \Rightarrow \quad d_f = 3 - \frac{0.42}{0.88} \simeq 2.55$$

$$For \quad d \geq 6: b = 1, n = 1/2 \quad \Rightarrow \quad d_f = 6 - \frac{1}{1/2} = 4$$

- d_f=4 for all $d \geq 6$
- d_c=6 is the upper critical dimension
- Same d_f is for finite clusters at $p \geq p_c$ and $p < p_c$

Percolation Chemical Dimension

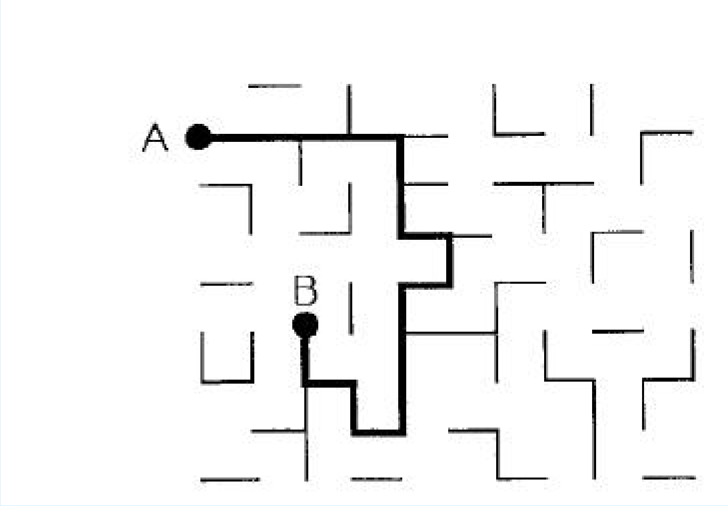
- The fractal dimension d_f is not enough to characterize the percolation cluster

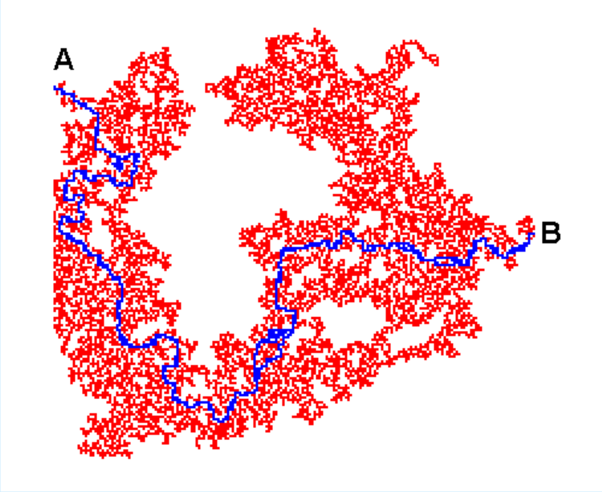


- This is obvious when looking on DLA and percolation
- For d=3 both have same $d_f \simeq 2.5$
- Percolation has loops – DLA has no loops

Shortest Path

➢ For better characterization we study the shortest path between A and B





➢ The shortest path ℓ is self-similar with fractal dimension d_{min}

$$\ell \sim r^{d_{min}}$$

Chemical Dimension

- The chemical dimension d_ℓ is defined by

$$M \sim \ell^{d_\ell} \sim r^{d_{\min} \cdot d_\ell}$$

Since $M \sim r^{d_f}$ $\Rightarrow$ $\boxed{d_f = d_{\min} \cdot d_\ell}$

- For percolation in $d = 2$ $\quad d_{\min} \simeq 1.13$

  $d = 3$ $\quad d_{\min} \simeq 1.38$

  Only numerical simulations ! No theory !

- For $d \geq 6$ $\quad d_{\min} = 2$

  Theory: in ∞ dimensions - no interactions
  each path is a random walk

  $\ell \sim r^2$ $\quad$ (ℓ - is like time)

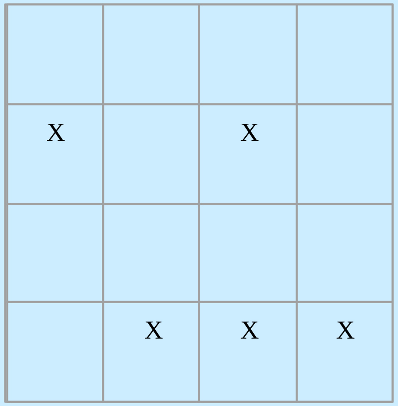
- $d_{\min}$ distinguish between DLA and percolation

  $d_{\min} = 1.38$ $\quad$ for percolation (d=3)

  $d_{\min} = 1$ $\quad$ for DLA (d=3)

Disease and fire spreading

- The chemical distance is important for describing disease and fire spreading
- Assume sick people or trees are on a lattice with concentration p
- At each stage one chemical shell is infected
- The total number of trees burned until time $t = \ell$ is

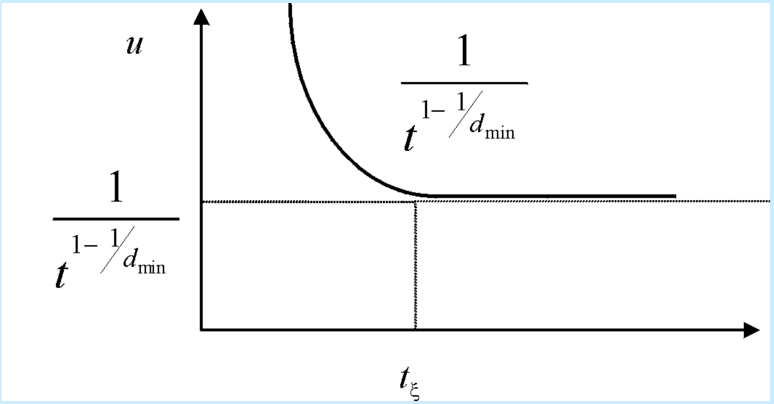


$$M(t) = t^{d_\ell}$$

- The distance where the disease or fire reached is

$$r \sim t^{1/d_{\min}}$$

- The velocity of spreading is



$$u = \frac{dr}{dt} \propto t^{\frac{1}{d_{\min}}-1}$$

- This is true only for $r \sim t^{1/d_{\min}} < x$

For $r > x$ $r \sim t$ (regular lattice)

$\Rightarrow$ $u =$ **constant**

- The constant velocity will occur at

$$r = x = (p - p_c)^{-n} \ and \ t_x \sim x^{d_{\min}} \sim (p - p_c)^{-n\,d_{\min}}$$

- We can calculate the constant velocity:

From $u \simeq t^{\frac{1}{d_{\min}}-1}$

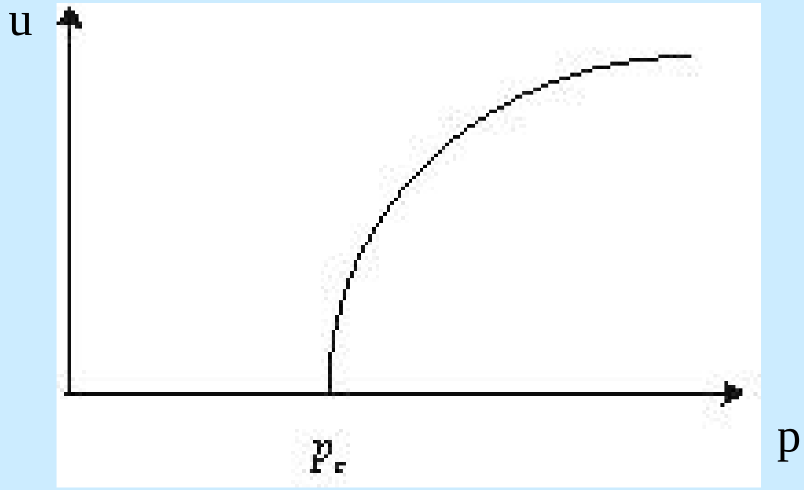
$$u \simeq t_x^{\frac{1}{d_{\min}}-1} \simeq (p-p_c)^{-n d_{\min}\left(\frac{1}{d_{\min}}-1\right)}$$

$$\boxed{u \simeq (p-p_c)^{(d_{\min}-1)n}}$$

- For $d=2$ $\quad (d_{\min}-1)n = 0.13 \cdot \frac{4}{3} \cong 0.16$

- very small exponent

- If at $p<p_c$ the fire (or disease) does not spread at all
just above p_c the fire (or disease) spreads very fast which increases when p increases

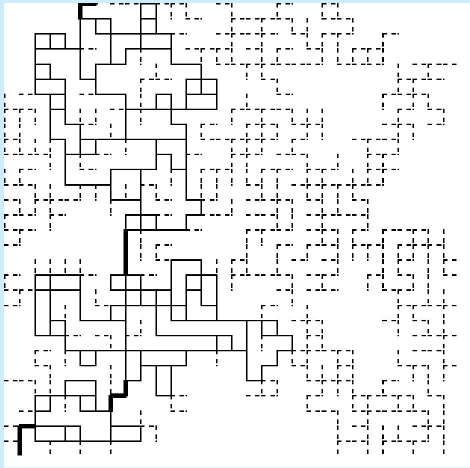


Percolation – substructures

Backbone, dead ends, red bond and blobs

- The fractal dimensions d_f and $d_{\min}$ are not enough to characterize percolation clusters

- We impose a voltage drop between two sites on the infinite cluster



- The backbone includes all bonds which carry current

- The dead ends are the parts that do not carry any current

- The red bonds (called also singly connected bonds) are those links that carry all current (dark black). When they are cut the current stops

- Blobs are the parts of the backbone left after removing the red bonds