Are scale-free networks self-similar?

Mean Distance in Scale Free Networks

$$P(k) \sim k^{-\lambda}$$

$$\ell = const. \qquad \lambda = 2$$

Ultra Small World

$$\ell = \log \log N \qquad 2 < \lambda < 3$$

$$\ell = \frac{\log N}{\log \log N} \qquad \lambda = 3$$

(Bollobas, Riordan, 2002)

Small World

$$\ell = \log N \qquad \lambda > 3$$

(Bollobas, 1985)
(Newman, 2001)

Cohen, Havlin   Phys. Rev. Lett. 90, 58701(2003)

Cohen, Havlin and ben-Avraham, in Handbook of Graphs and Networks

eds. Bornholdt and Shuster (Willy-VCH, NY, 2002) chap.4

Confirmed also by: Dorogovtsev et al (2002), Chung and Lu (2002)

Are scale-free networks self-similar?

- On one hand scale-free – no characteristic degree suggest self-similarity--FRACTALITY

- On the other hand, for self-similarity or invariance under length scale transformation one needs a power-law relation

$$N = \ell^{d_\ell}$$

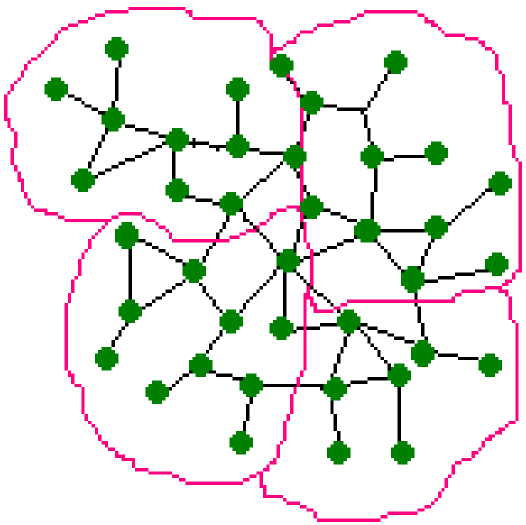
Since it must be a solution of: $N(bL) = b^{d_f} N(L)$

- Here $\ell = \log N$ or $\ell = \log \log N$

Thus, $N = e^{\ell / \ell_0}$ or $N = e^{e^{\ell / \ell_0}}$

Small World or Ultra-small World are Against Self-Similarity!

Box counting method



- Generate boxes where all nodes are within a distance ℓ
- Calculate number of boxes, $n(\ell)$, of size ℓ needed to cover the network
- We obtain for WWW, social networks, cellular networks, etc.

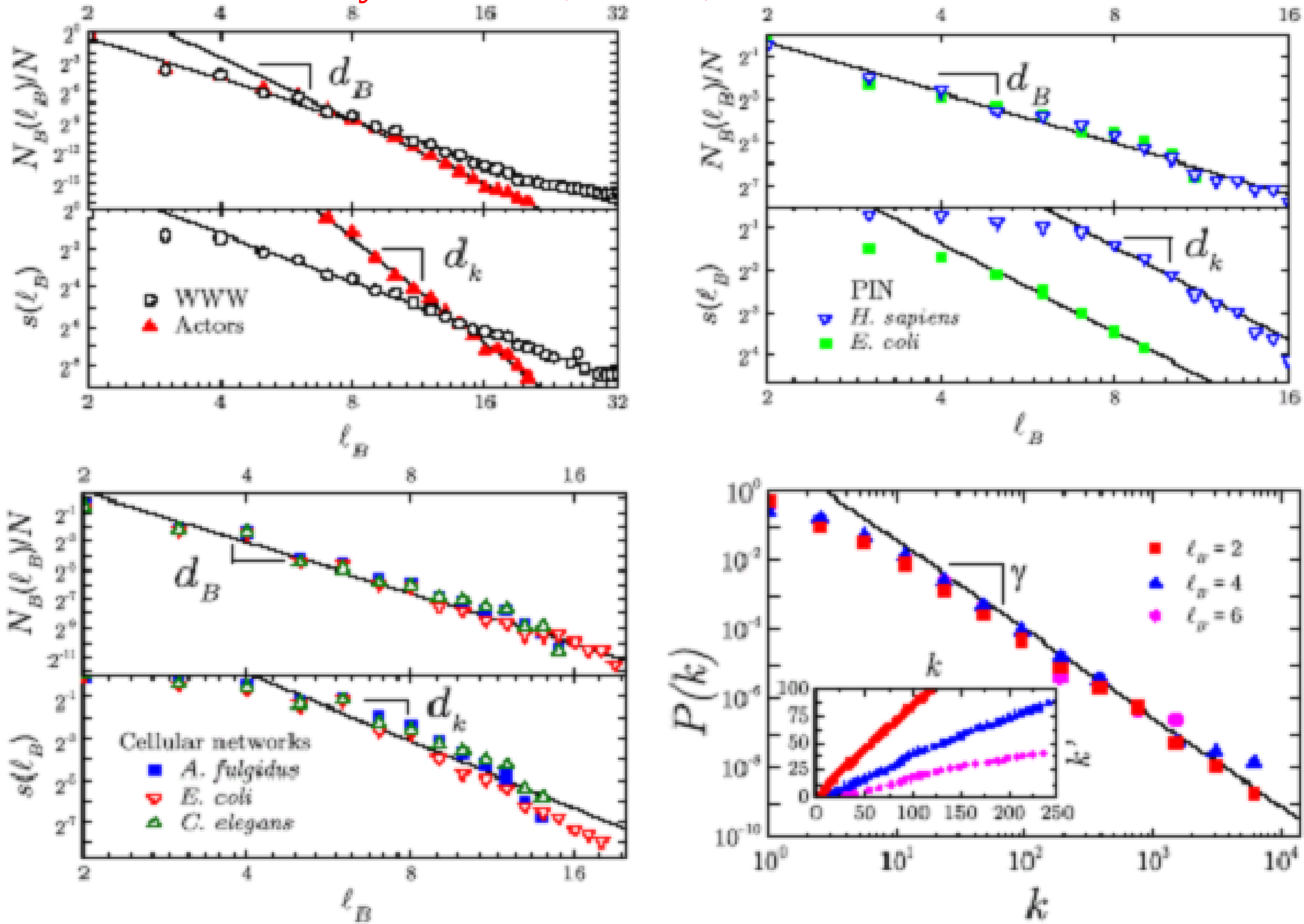
$$N_B(\ell) \sim \ell^{-d_\ell}$$

or

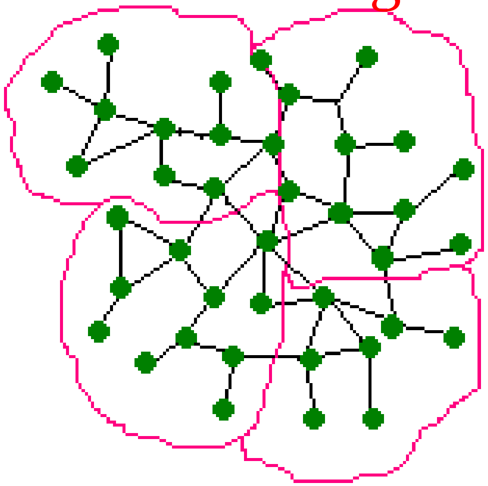
$$N \sim \ell^{d_\ell} \qquad 2 < d_\ell < 5$$ ⇨ Self similarity

How can one reconcile this and the exponential relation with distance?

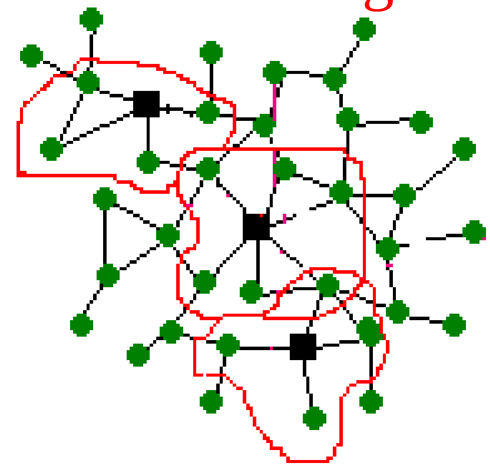
Self similarity of WWW, Actors, PIN and Cellular Networks

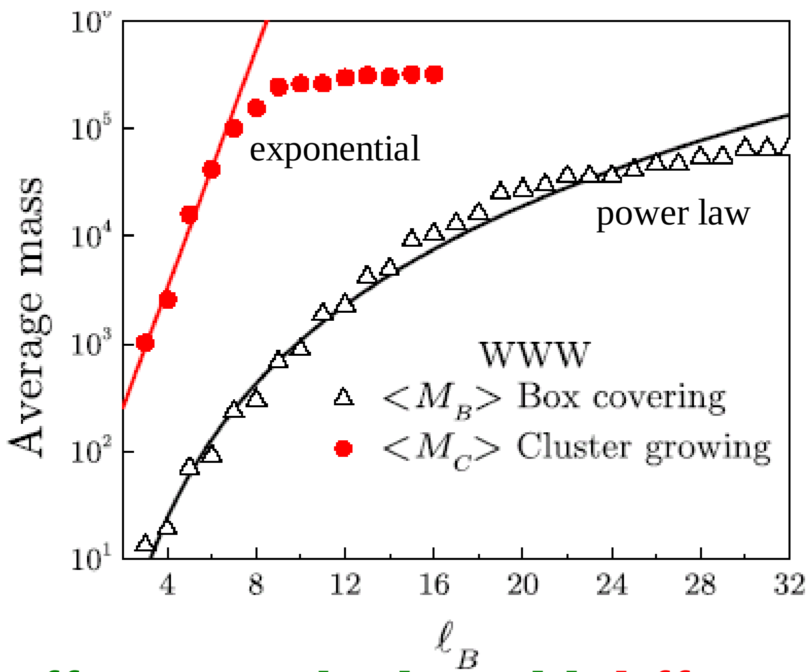


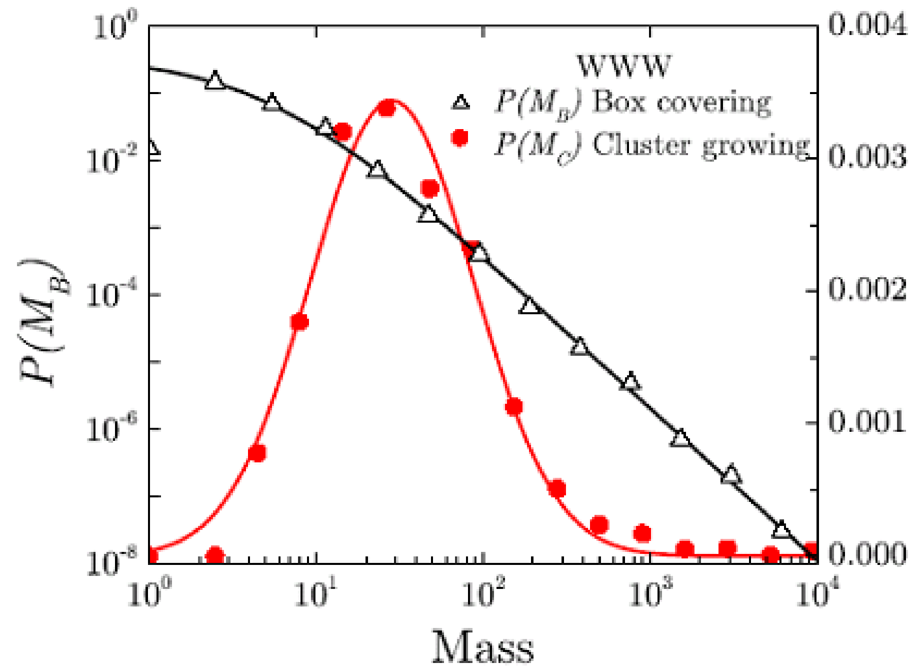
Box Covering



Cluster Growing







Different methods yield different results due to heterogeneous topology

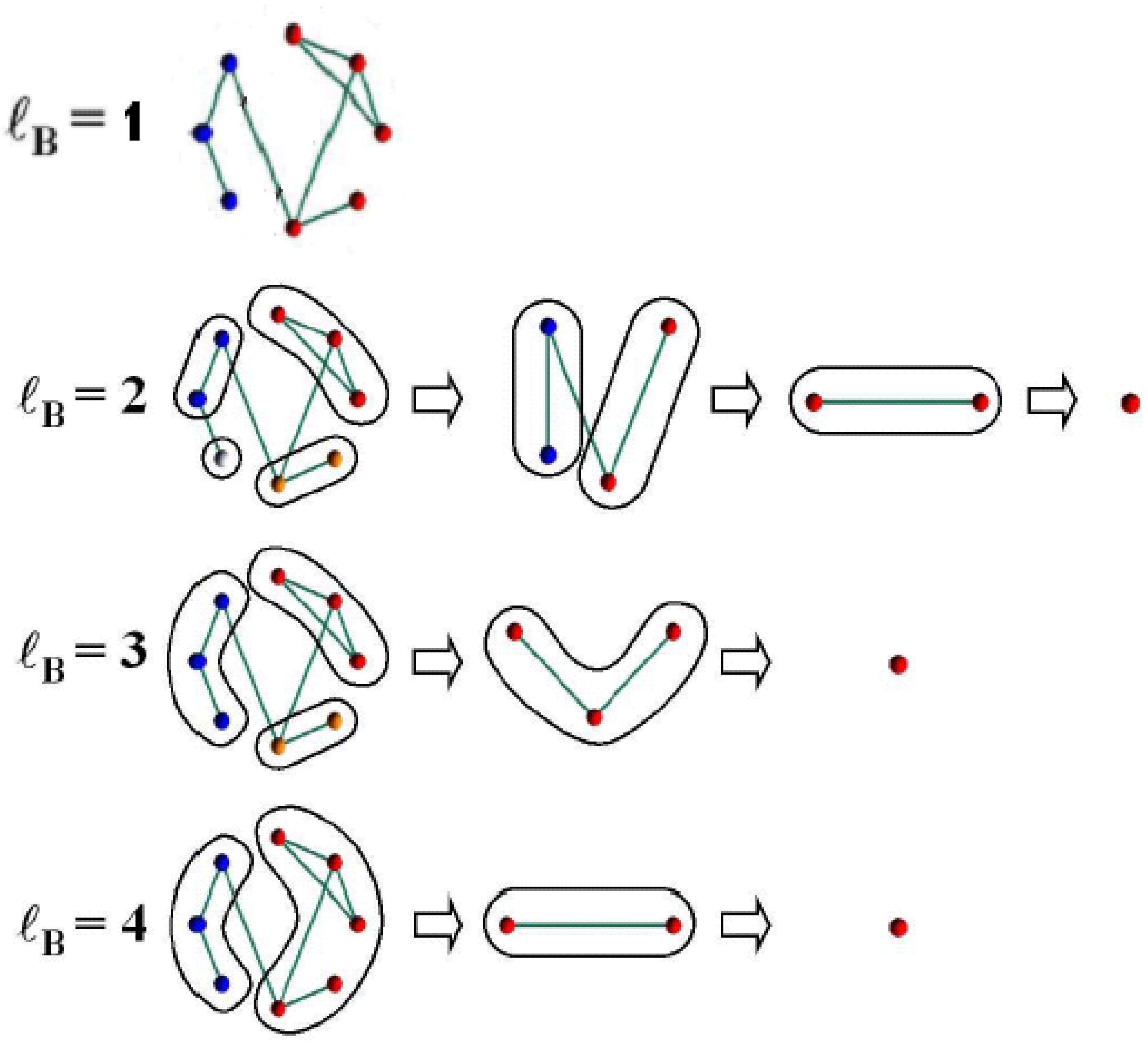
Box covering reveal the self similarity!! Cluster growth reveal the small world!!

NO CONTRADICTION!!! SAME HUBS ARE USED MANY TIMES FOR SW

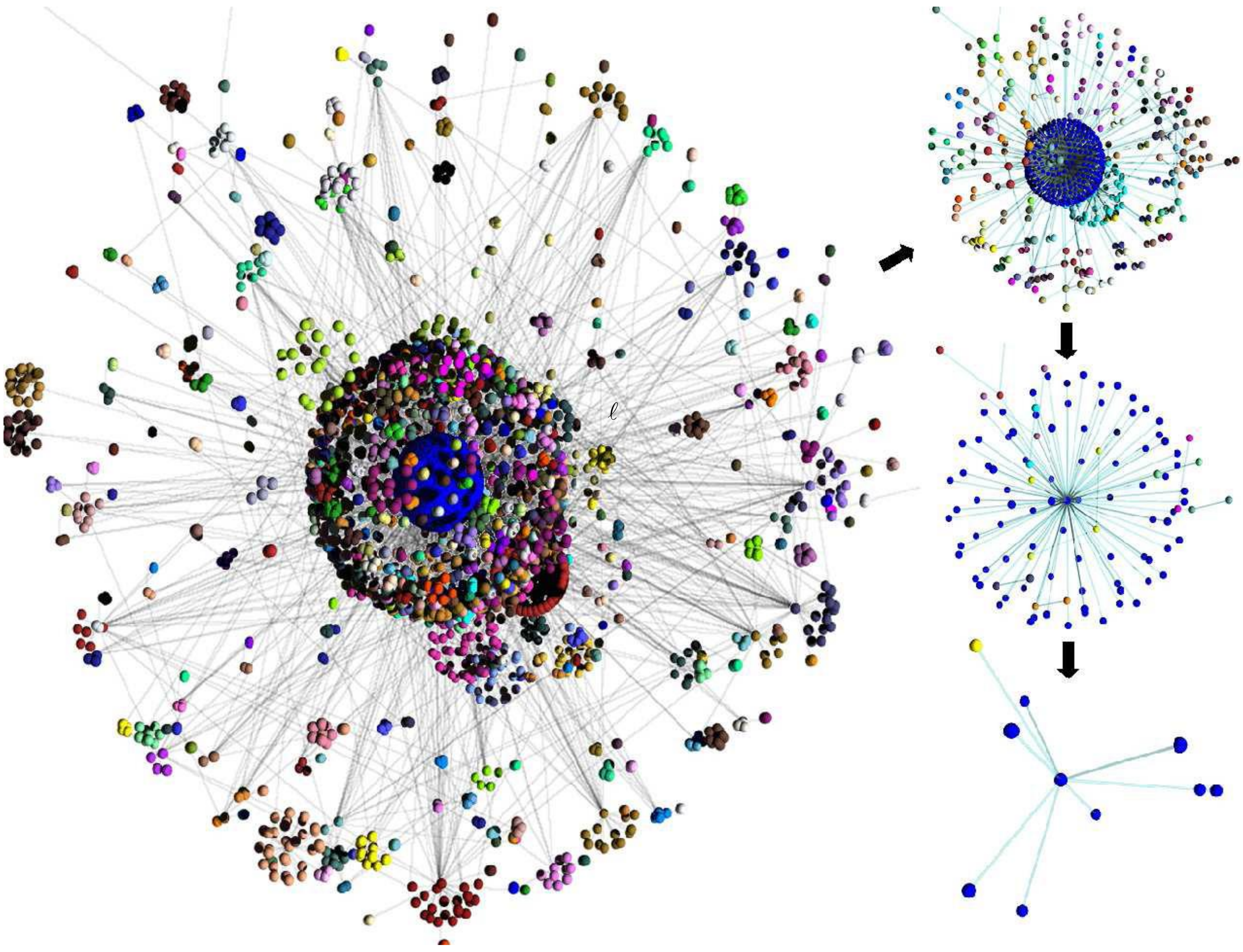
Hierarchy of Scale Free

Renormalization and Box Covering Approach

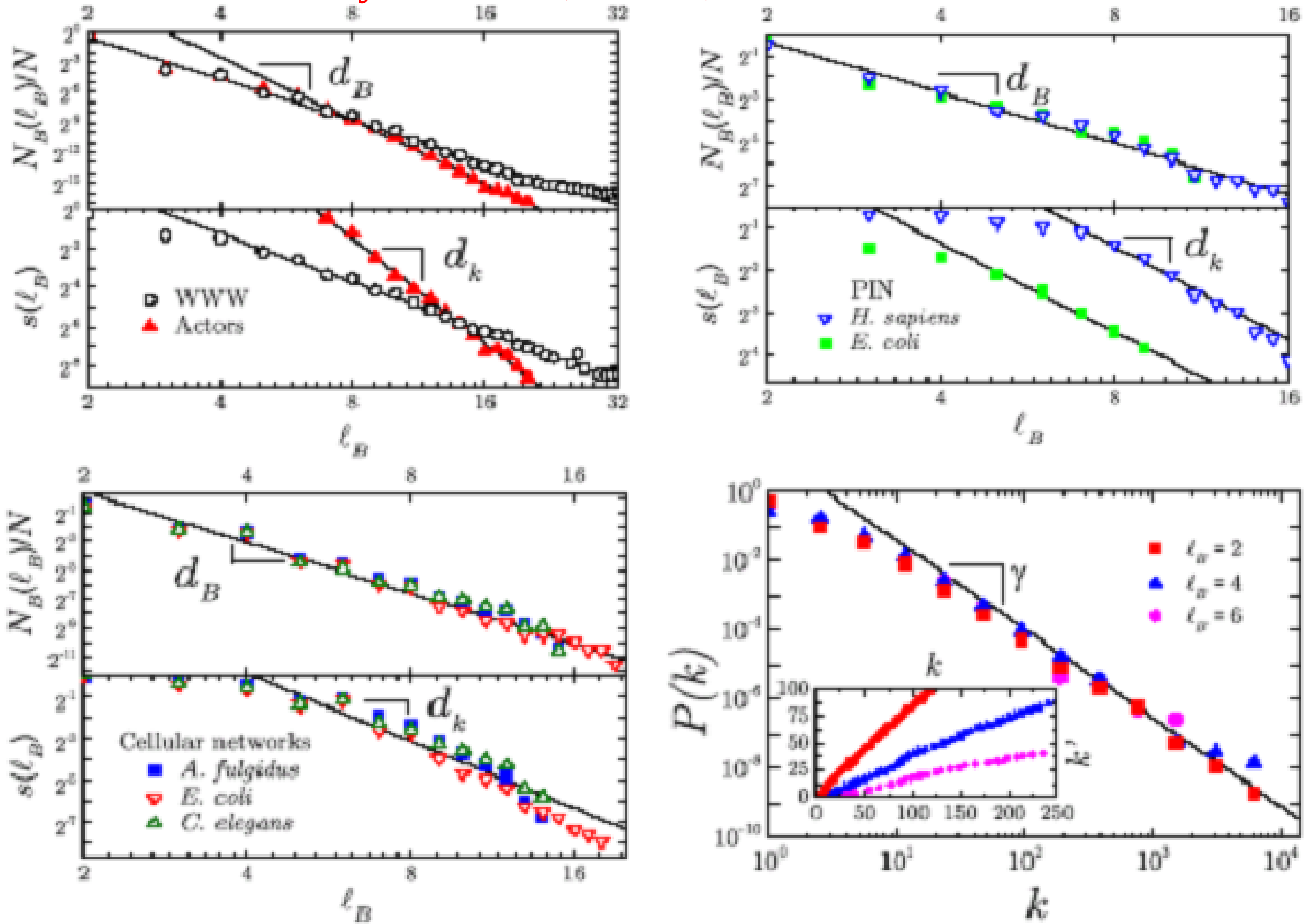
NOW REGARD
EACH BOX AS
A SINGLE NODE
AND ASK WHAT
IS THE DEGREE
DISRIBUTION OF
THE NETWORK
OF BOXES AT
DIFERENT
SCALES ?



Renormalization of WWW network with $\ell_B = 3$



Self similarity of WWW, Actors, PIN and Cellular Networks



Hierarchy of Scale Free

After Renormalization: $P(k) \rightarrow P'(k') \sim (k')^{-\lambda}$

With the same λ !

Where $k \rightarrow k' = s(\ell_B)k$

THE SCALING TRANSFORMATION OF THE DEGREE DISTRIBUTION

$$s(\ell_B) \sim \ell_B^{-d_k}$$

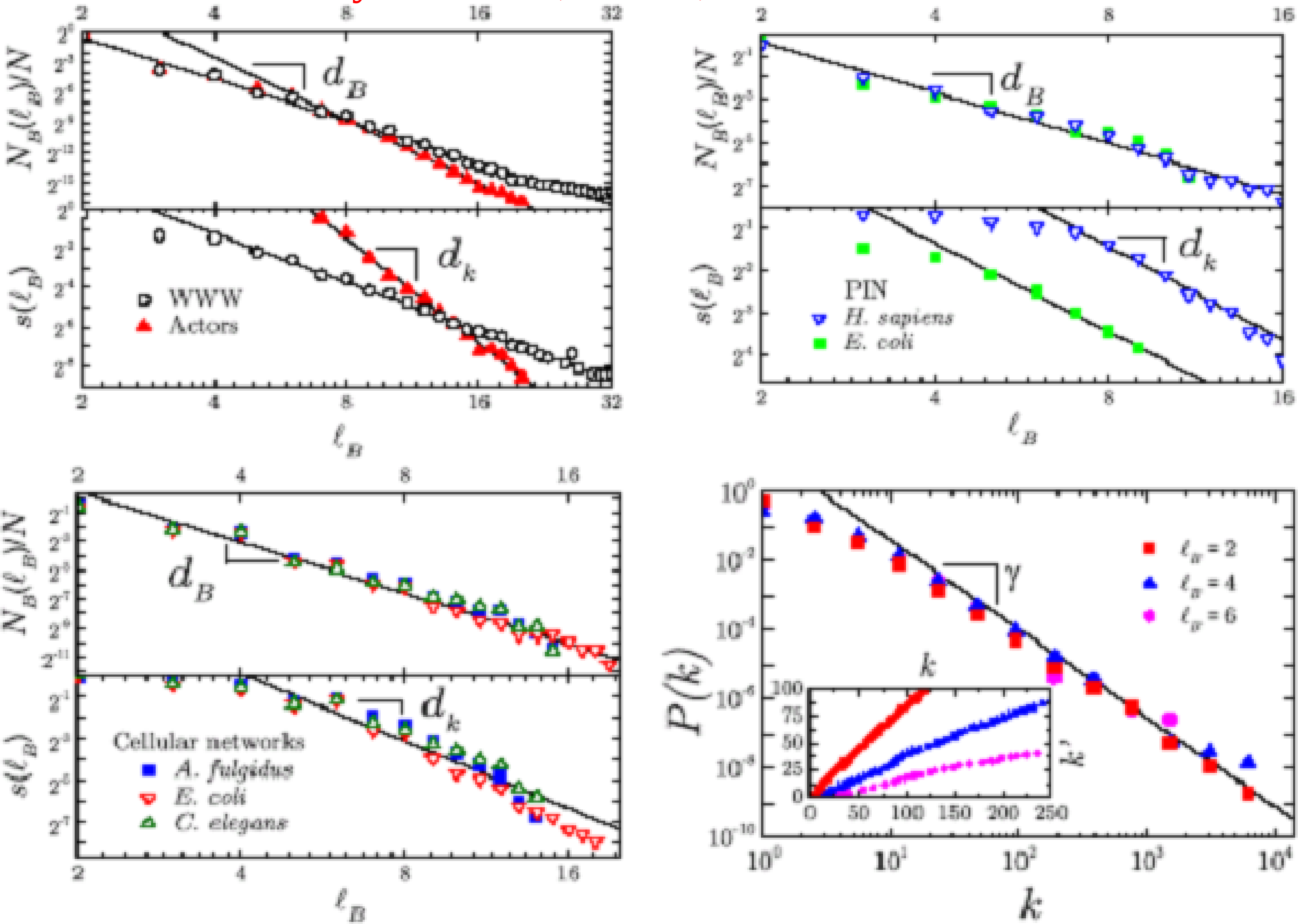
HOW FAMILIES OF VARIOUS SIZES ARE LINKED?

From which follows:

$$\lambda = 1 + \frac{d_B}{d_k}$$

Chaoming Song, SH, Hernan Makse, Nature, in press (2005)

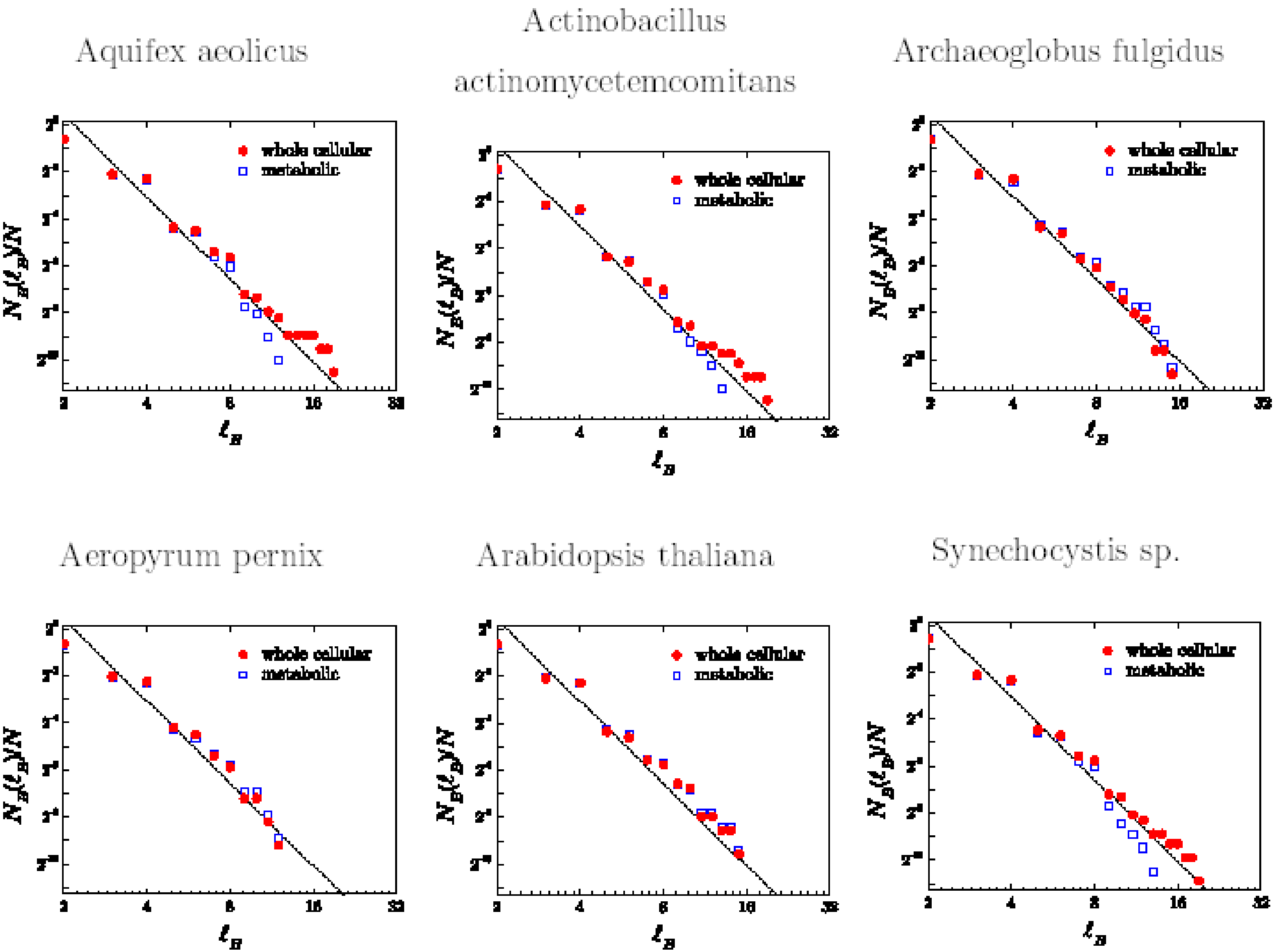
Self similarity of WWW, Actors, PIN and Cellular Networks



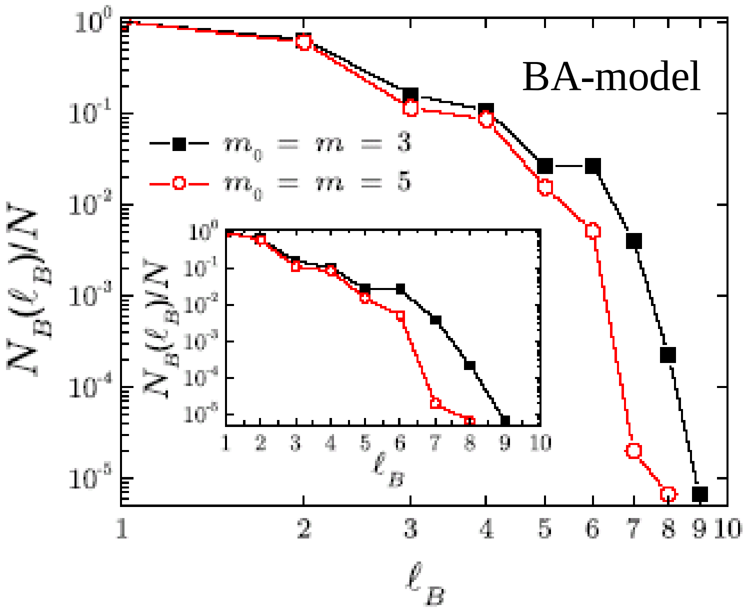
Fractal and Degree exponents for Various Networks

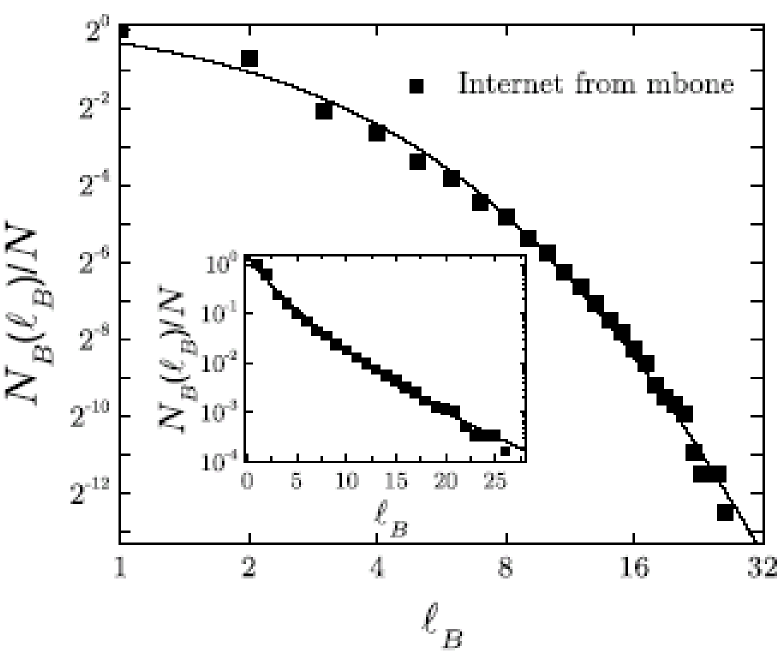
Network	d_B	d_k	$1 + d_B/d_k$	λ
WWW	4.1	2.5	2.6	2.6
Actor	6.3	5.3	2.2	2.2
E. coli (PIN)	2.3	2.1	2.1	2.2
H. sapiens (PIN)	2.3	2.2	2.0	2.1
43 cellular networks	3.5	3.2	2.1	2.2
Scale-free tree	3.4	2.5	2.4	2.3

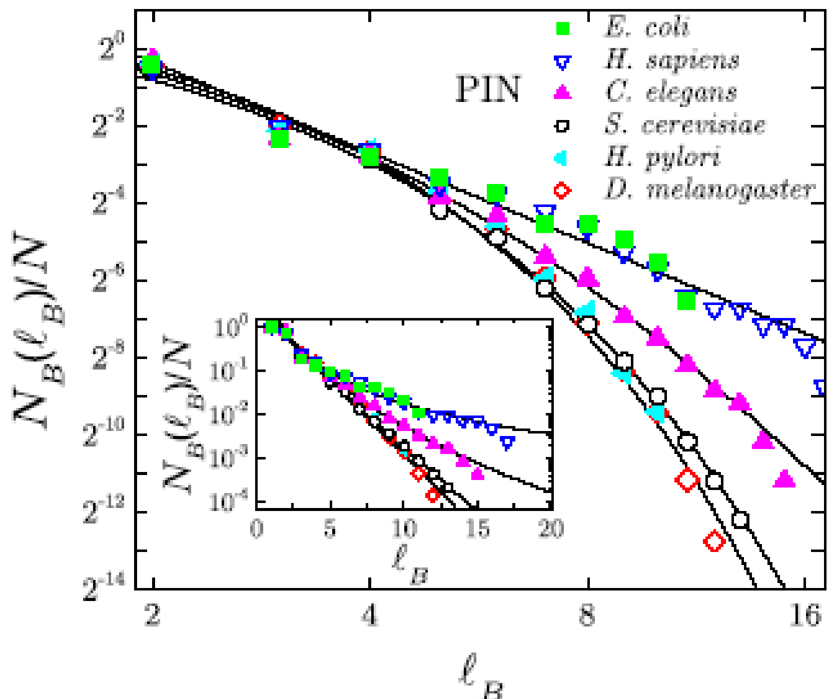
TABLE I: Summary of the exponents obtained for the scale-invariant networks studied in the manuscript.


Aquifex aeolicus
Actinobacillus actinomycetemcomitans
Archaeoglobus fulgidus
Aeropyrum pernix
Arabidopsis thaliana
Synechocystis sp.
whole cellular
metabolic
$N_B(\ell_B)/N$
ℓ_B
2
4
8
16
32

NOT ALL REAL NETWORKS ARE FRACTALS!
ALMOST NO MODEL!!

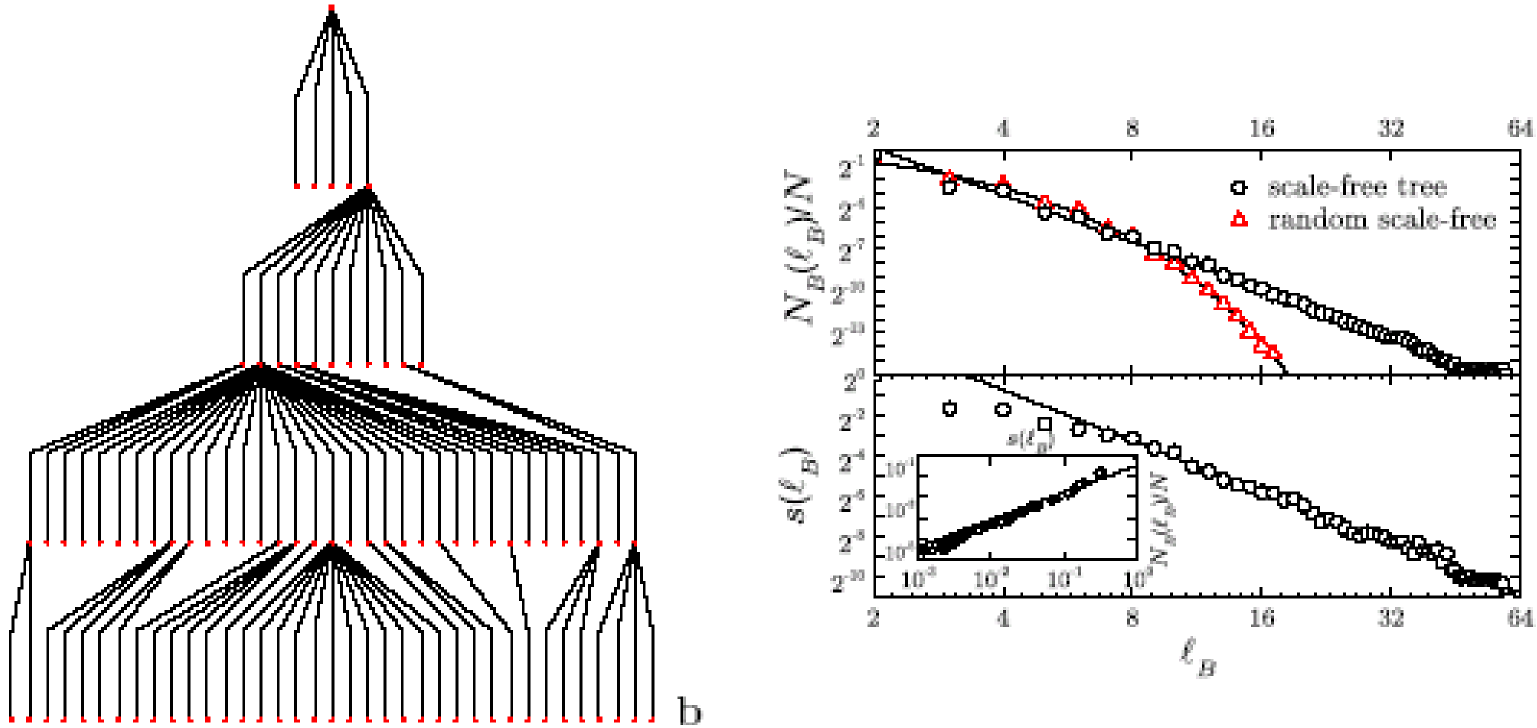


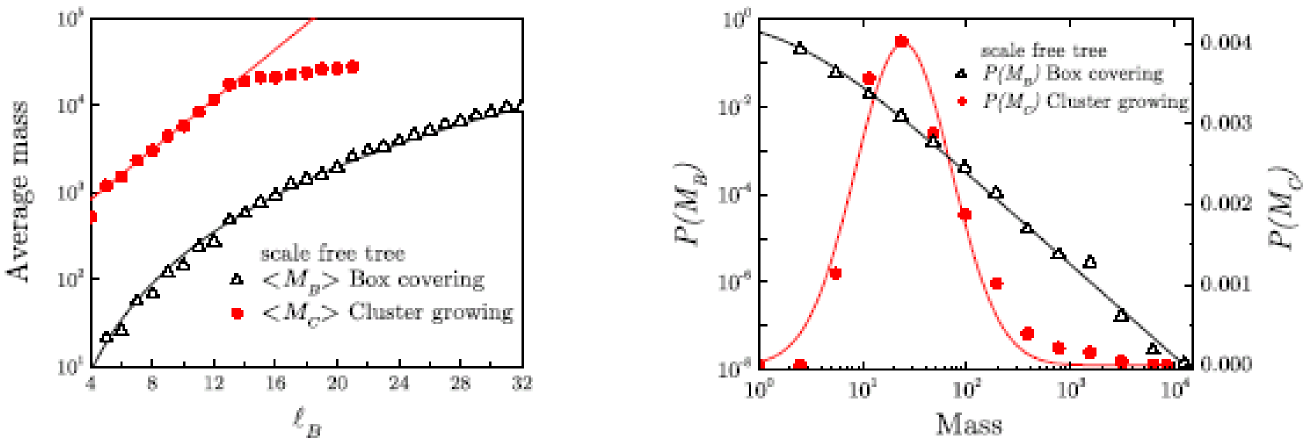




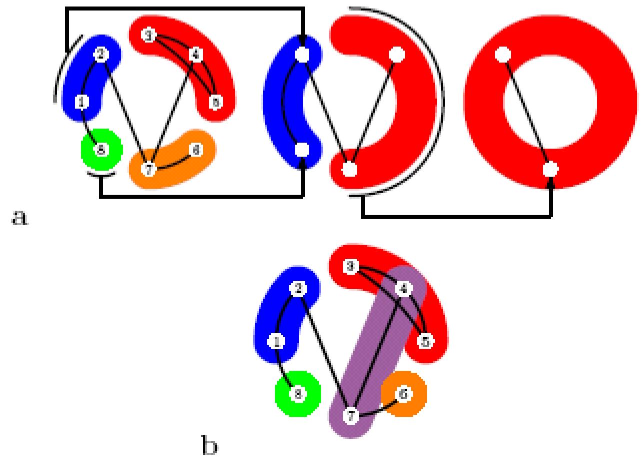
Summary and Challenges

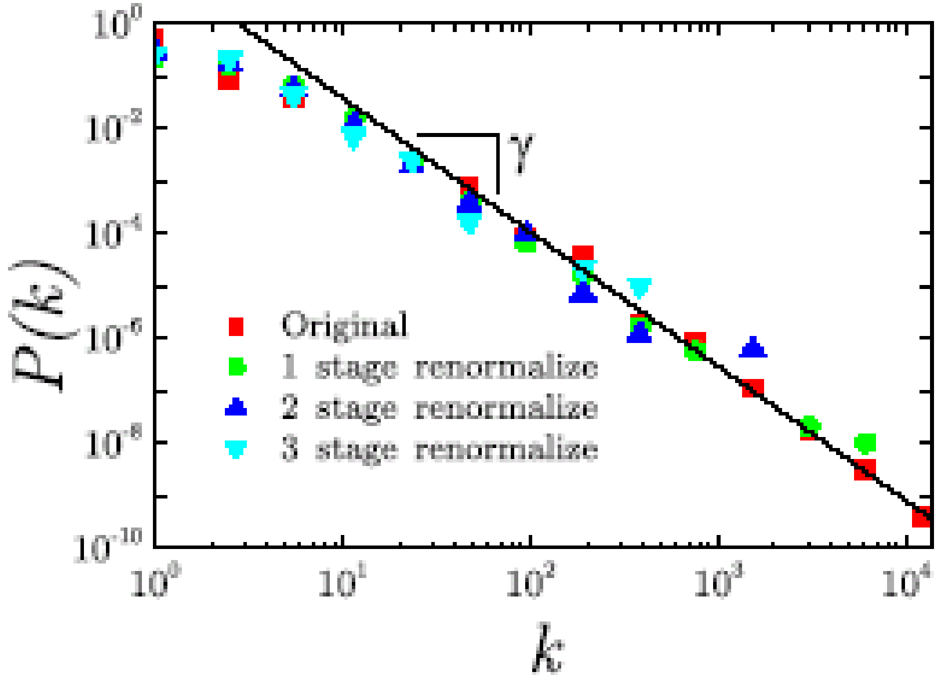
- In contrast to common believe, many real world networks are self similar, $M_B \sim \ell_B^{d_B}$
- The degree distribution is scale free under length scale transformation, $P(k) \approx k^{-\lambda}$
- The scaling of degree distribution $k' = s(\ell_B)k$ where $s(\ell_B) \sim \ell_B^{-d_k}$
- Finally: $\lambda = 1 + \dfrac{d_B}{d_k}$
- What is the origin of self similarity?
- Most models are not self similar!! How to generate a self similar (fractal) network?

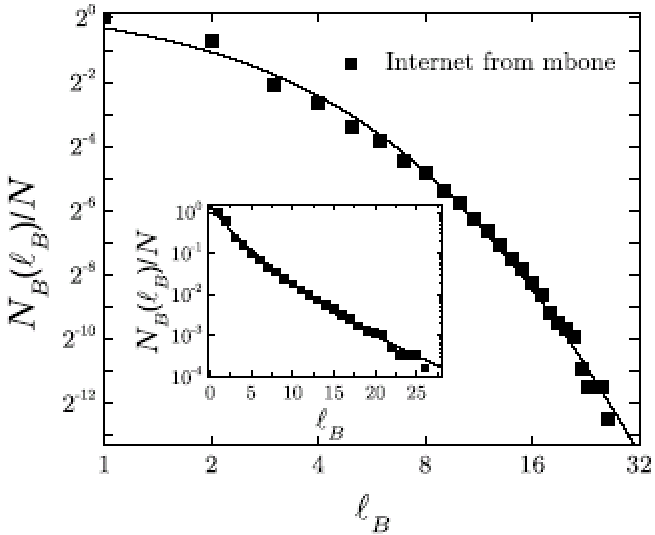

a
b
scale-free tree
random scale-free
$N_B(\ell_B)/N$
$s(\ell_B)$
ℓ_B
2
4
8
16
32
64

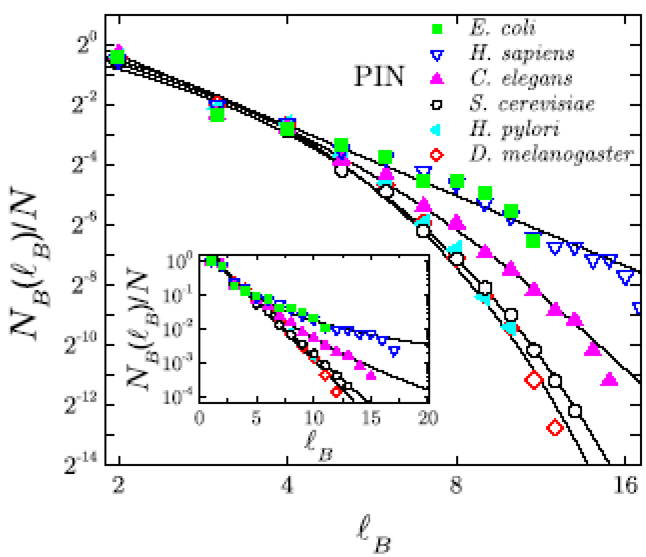

Average mass
ℓ_B
scale free tree
$<M_B>$ Box covering
$<M_C>$ Cluster growing
$P(M_B)$
$P(M_C)$
Mass
scale free tree
$P(M_B)$ Box covering
$P(M_C)$ Cluster growing

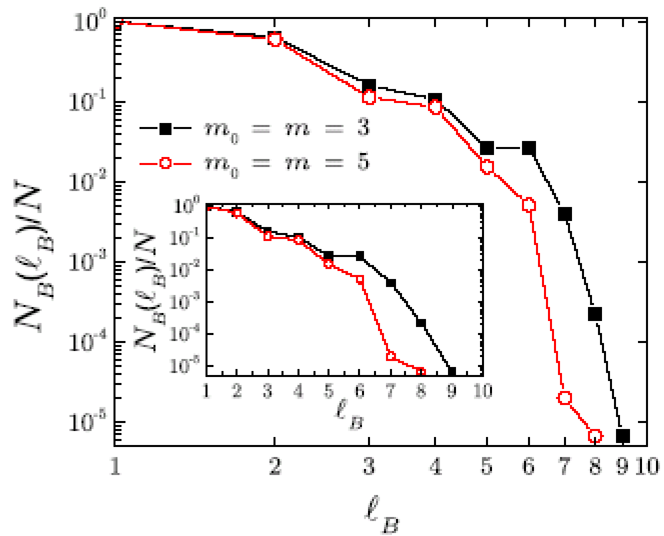
a b


a
b

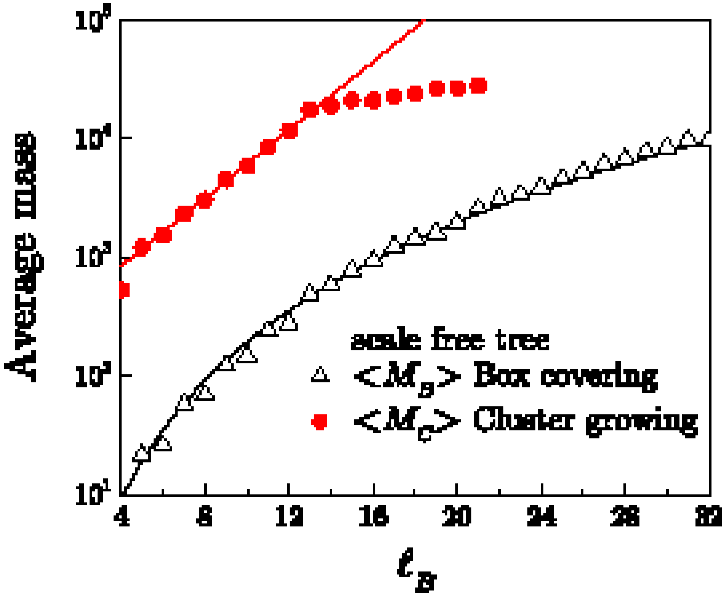

P(k)
k
γ
Original
1 stage renormalize
2 stage renormalize
3 stage renormalize

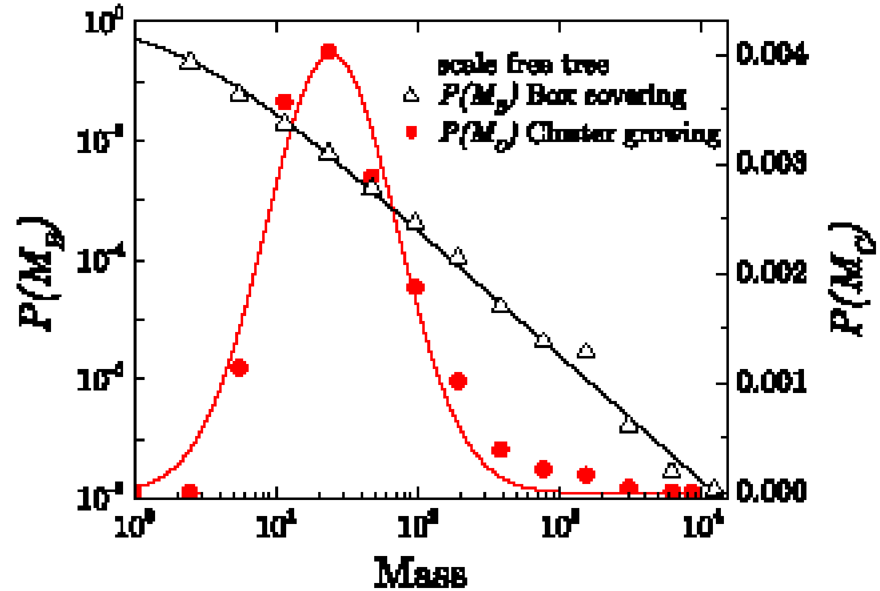

Internet from mbone
$N_B(\ell_B)/N$
ℓ_B

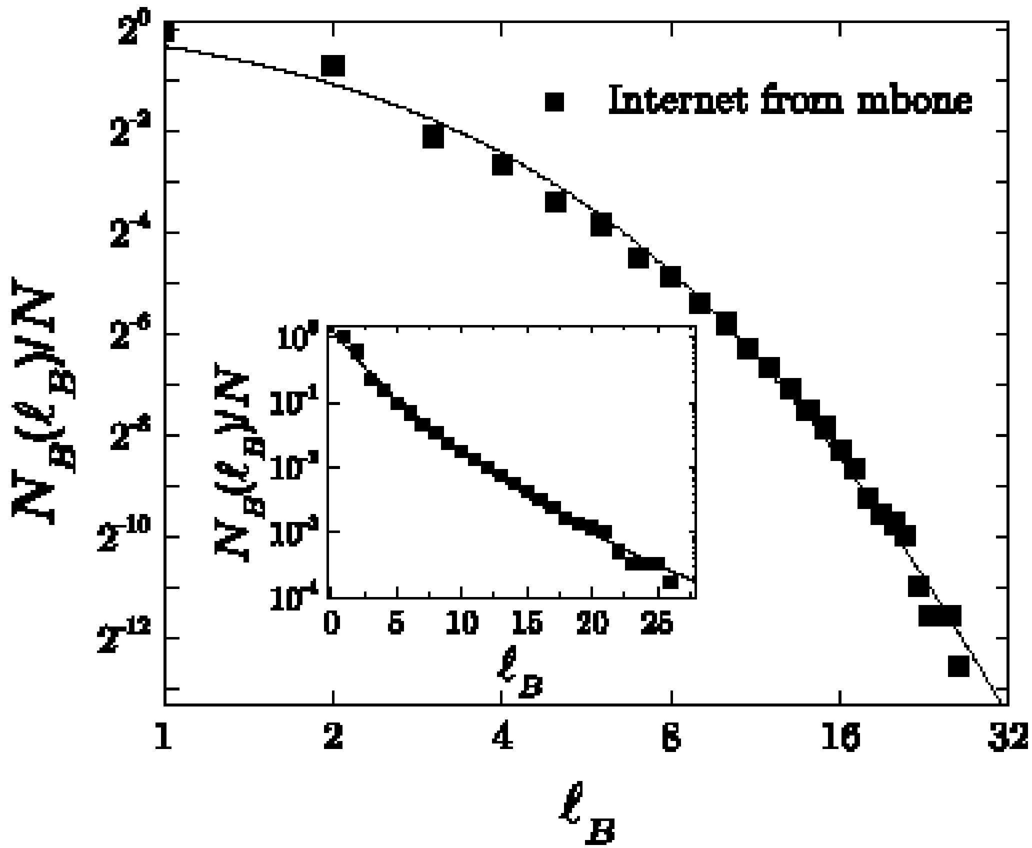

PIN
E. coli
H. sapiens
C. elegans
S. cerevisiae
H. pylori
D. melanogaster
$N_B(\ell_B)/N$
ℓ_B



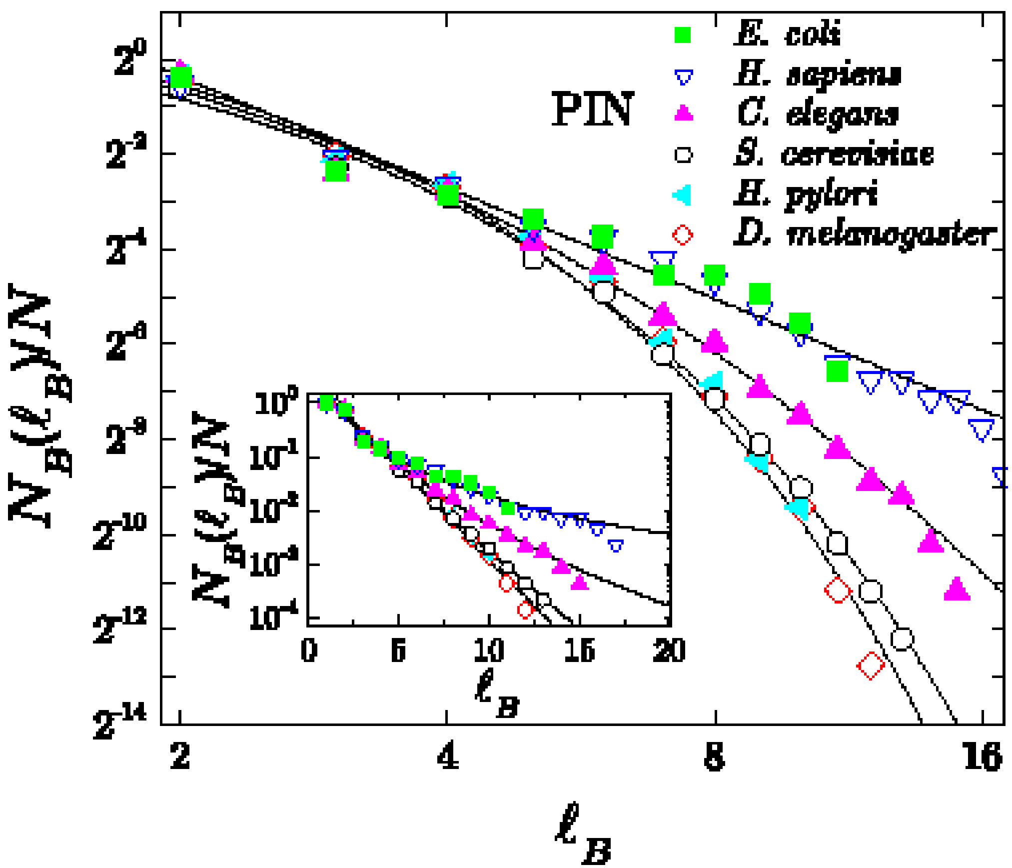
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$N_B(\ell_B)/N$
$m_0 = m = 3$
$m_0 = m = 5$
1 2 3 4 5 6 7 8 9 10
ℓ_B


Average mass
ℓ_B
scale free tree
<M_B> Box covering
<M_C> Cluster growing
4
8
12
16
20
24
28
32

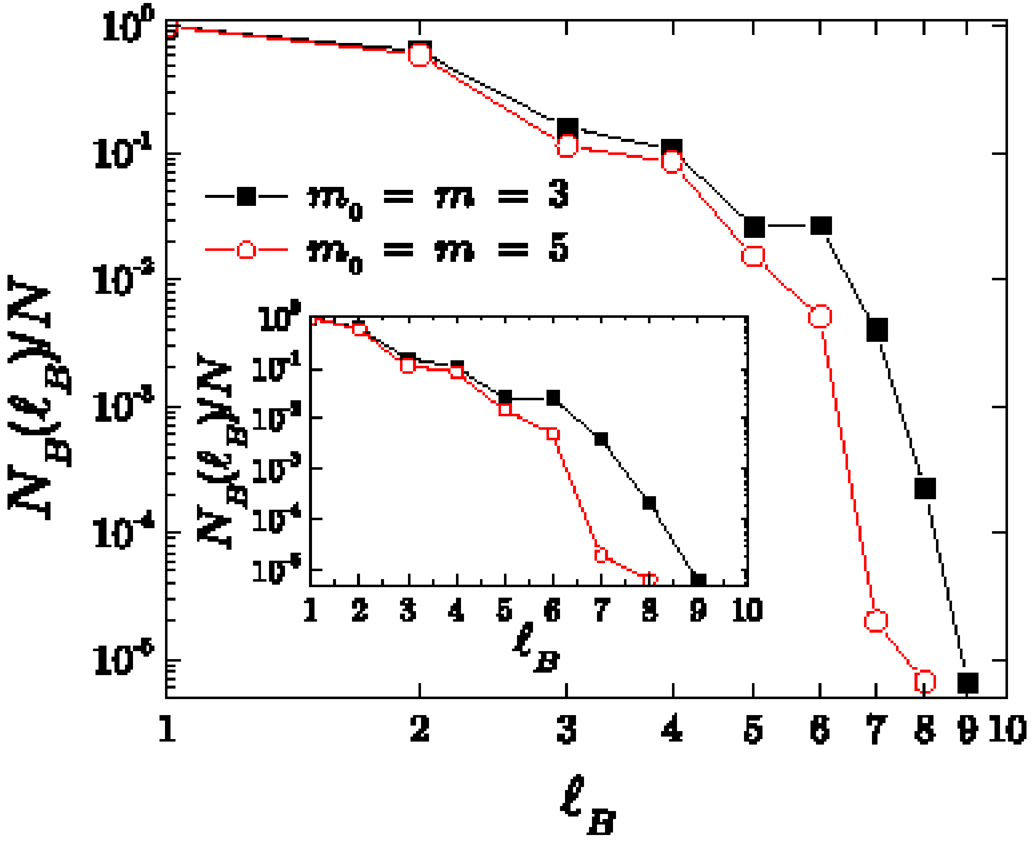

$P(M_B)$
$P(M_C)$
Mass
scale free tree
$P(M_B)$ Box covering
$P(M_C)$ Cluster growing
0.004
0.003
0.002
0.001
0.000



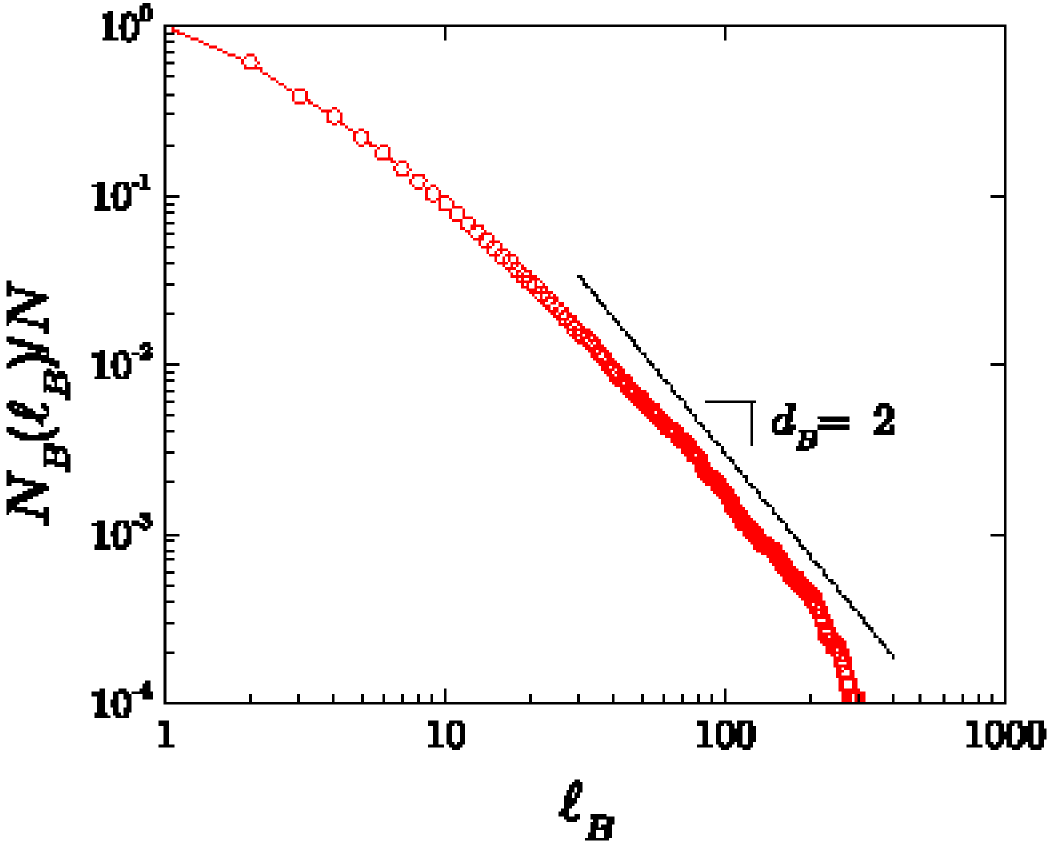
Internet from mbone
$N_B(\ell_B)/N$
ℓ_B



PIN
E. coli
H. sapiens
C. elegans
S. cerevisiae
H. pylori
D. melanogaster
$N_B(\ell_B)/N$
ℓ_B
2^0
2^{-2}
2^{-4}
2^{-6}
2^{-8}
2^{-10}
2^{-12}
2^{-14}
2
4
8
16
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
5
10
15
20



$N_B(\ell_B)/N$
$m_0 = m = 3$
$m_0 = m = 5$
ℓ_B
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
1
2
3
4
5
6
7
8
9
10



10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$N_B(\ell_B)/N$
$d_B = 2$
1
10
100
1000
ℓ_B