

## Angle restriction enhances synchronization of self-propelled objects

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Understanding the synchronization process of self-propelled objects is of great interest in science and technology. We propose a synchronization model for a self-propelled objects system in which we restrict the maximal angle change of each object to  $\theta_R$ . At each time step, each object moves and changes its direction according to the average direction of all of its neighbors (including itself). If the angle change is greater than a cutoff angle  $\theta_R$ , the change is replaced by  $\theta_R$ . We find that (i) counterintuitively, the synchronization improves significantly when  $\theta_R$  decreases, (ii) there exists a critical restricted angle  $\theta_{Rc}$  at which the synchronization order parameter changes from a large value to a small value, and (iii) for each noise amplitude  $\eta$ , the synchronization as a function of  $\theta_R$  shows a maximum value, indicating the existence of an optimal  $\theta_R$  that yields the best synchronization for every  $\eta$ .

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### I. INTRODUCTION

Over the past 10 years, there has been widespread scientific interest in the collective motion exhibited by such groups as a school of fish, a flock of birds, or a swarm of robots [1–11]. Collective motion due to synchronization processes (called also polarization or orientational ordering of velocity) plays an important role in many different fields, including biology, ecology, climatology, sociology, and technology, and is even a factor in the arts [12–18]. A surprisingly simple but useful model, proposed by Vicsek *et al.* [19], shows the existence of phase transition in a self-propelled objects (SPO) system. More recently, various specialized models have been proposed to describe, e.g., accelerating convergence [20], avoiding collision [21,22], and enhancing the efficiency of convergence [23]. Also, other properties of SPO such as coherence [24] and collective intelligence [25] were studied. These in turn have promoted the development of such applications as distributed sensor networks [26], unmanned aerial vehicles [27], underwater vehicles [28], and altitude alignment of satellite clusters [29]. However, one of the most important issues, in particular for applications, is the question as to how to improve synchronization of SPO based on local information. Even more important is to identify optimal synchronization conditions in the presence of noise, which exists in a group of SPO machines such as robots or vehicles.

In this paper, we develop a restricted angle SPO model (RASPO), which generalizes the Vicsek model (VM) [19] and counterintuitively improves the synchronization of SPO systems. Specifically, we find that (i) the synchronization is significantly improved when the restricted angle decreases; (ii) there exists a critical restricted angle  $\theta_{Rc}$  above which the synchronization order parameter changes from a large value to a small value; and (iii) for a given noise amplitude  $\eta$ , the synchronization shows a peak as a function of  $\theta_R$ , which yields the best synchronization conditions.

### II. THE ANGLE RESTRICTION MODEL

In the VM, a group of  $n$  objects move in a  $L \times L$  square with the same constant speed in different directions. Initially,

the objects are randomly distributed, and their directions are also uniformly randomly distributed in the interval  $(0, 2\pi)$ . At each time step, the direction of each object is determined by the average directions of all the objects within a circle centered at the given object, the influencing radius of which is  $R$ . At time  $t$ , the position of a specific object is updated according to

$$x_i(t+1) = x_i(t) + v_0 e^{i\theta_i(t)}. \quad (1)$$

Its direction is updated as

$$e^{\theta_i(t+1)} = e^{i\Delta\theta_i(t)} \frac{\sum_{j \in \Gamma_i(t+1)} e^{i\theta_j(t)}}{\left\| \sum_{j \in \Gamma_i(t+1)} e^{i\theta_j(t)} \right\|}, \quad (2)$$

where  $\|\cdots\|$  is the standard norm [30] defined as  $\|(z_1, z_2, \dots, z_n)\| = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{1/2}$ ,  $\Delta\theta_i \in [-\eta, \eta]$  denotes the white noise,  $e^{i\theta_i(t)}$  is a unit directional vector, and  $\Gamma_i(t+1)$  is the set of neighbors of object  $i$  at time step  $t+1$ . In order to measure the synchronization of the system, an order parameter is introduced as [19,31]

$$V_\alpha = \frac{1}{n} \left\| \sum_{i=1}^n e^{i\theta_i(t)} \right\|, \quad 0 \leq V_\alpha \leq 1. \quad (3)$$

A larger value of  $V_\alpha$  indicates a better consensus, and when  $V_\alpha = 1$ , all the objects are moving in the same direction. Numerical simulations show that when the density is high and the noise low, all the objects will have reached consensus, i.e., will be moving in the same direction after a finite number of time steps (the convergence time) [32,33].

In the VM and all related models of SPO, the rotation and rectilinear motions of each object can be treated separately. Each object changes direction to the average direction of its neighbors. It then moves a distance  $v_0$  rectilinearly.

However, because the directions and positions of all the objects are initially randomly distributed, most of the objects make sharp changes in direction that bear little similarity to behavior found in nature, and are thus impractical when developing applications in engineering. The movie supplied in Ref. [34] clearly demonstrates that a flying bird can not execute sharp changes in direction within a single step. From

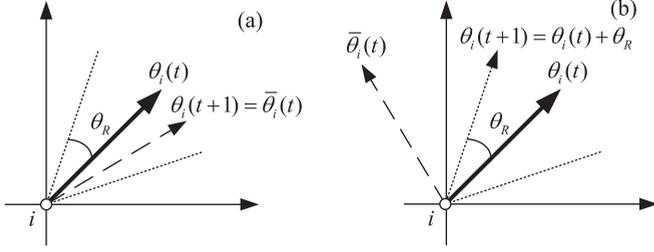


FIG. 1. Description of two cases of direction updated in the RASPO model: (a) the change from  $\theta_i(t)$  to  $\bar{\theta}_i(t)$  is smaller than  $\theta_R$ , so  $\theta_i(t+1) = \bar{\theta}_i(t)$ ; (b) the change from  $\theta_i(t)$  to  $\bar{\theta}_i(t)$  is greater than  $\theta_R$  and  $\bar{\theta}_i(t)$  is closer to  $\theta_i(t) + \theta_R$ , so  $\theta_i(t+1) = \theta_i(t) + \theta_R$ . Note that  $\theta_R$  is defined as half of the restricted angle and, therefore,  $\theta_R = \pi$  represents no angle restriction as in Vicsek's model.

the point of view of an engineer, any robot or vehicle powered by an engine can not make an acute-angle turn in a very short time period. In order to more closely resemble behavior found in nature and to be useful in developing real-world applications, we introduce a restricted angle model for an SPO system and find that this restriction dramatically improves the synchronization properties.

Figure 1 describes the RASPO model in which each object with direction  $\theta_i(t)$  updates its direction and position within a radius  $R$ . The model (i) calculates an average for the directions of all its neighbors  $\bar{\theta}_i(t)$ , (ii) calculates the changes from  $\theta_i(t)$  to  $\bar{\theta}_i(t)$ , and determines whether it is smaller [Fig. 1(a)] or greater [Fig. 1(b)] than  $\theta_R$  [if the change is smaller, the result is  $\theta_i(t+1) = \bar{\theta}_i(t)$  and if it is greater, the result is  $\theta_i(t+1) = \theta_i(t) + \theta_R$ ], and (iii) each object then updates its position according to Eq. (1). (For a rigorous mathematical description of the RASPO model, see Appendix.) The RASPO model, which generalizes the VM, has six main parameters  $n$ ,  $L$ ,  $R$ ,  $v_0$ ,  $\eta$ , and  $\theta_R$ , each of which affects the synchronization differently. In order to study the different effects, we perform computer simulations of RASPO in which the density is fixed [3] at  $\rho = n/L^2 = 1$  with periodic boundary conditions [19].

### III. SIMULATION AND DISCUSSION

To investigate the main effect of introducing a restricted angle, we perform computer simulations of synchronization as a function of restricted angle  $\theta_R$  for different  $n$ , and for several values of  $R$  and  $v_0$  (see Fig. 2). Our simulation results show that synchronization increases significantly as  $\theta_R$  decreases. This is counterintuitive since one would assume that giving more freedom to the objects would facilitate greater synchronization. It is clear that when  $\theta_R$  is small, synchronization is high, while for  $\theta_R = \pi$ , where the model corresponds to the original VM, the synchronization is relatively low [35]. A plausible explanation for this finding is as follows. It seems that, even without any input of external noise, there exists some internal noise in the system, which is larger when  $\theta_R$  is larger. Therefore, reducing  $\theta_R$  is effectively like reducing the internal noise, which improves synchronization. Support for this conjecture can be found from Fig. 3. From Fig. 3(a), we can see that inherent noise is increasing with noise  $\eta$ , and from Fig. 3(b), we can also see that inherent noise is increasing with noise  $\theta_R$ . So, decreasing the restricted angle  $\theta_R$  may be regarded as

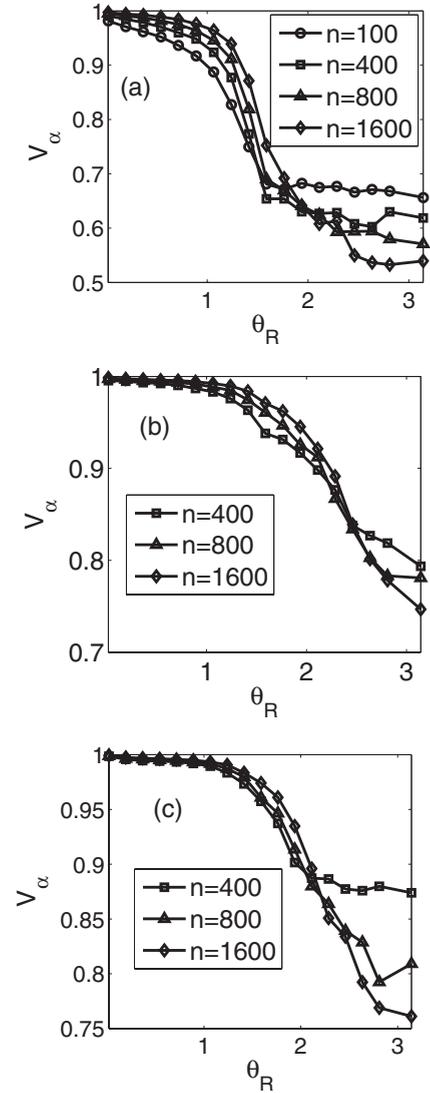


FIG. 2. The synchronization  $V_\alpha$  as a function of restricted angle  $\theta_R$  for different system size  $n$ . In the simulation, the density  $\rho = 1$ , (a)  $R = 0.3$  and  $v_0 = 0.1$ , (b)  $R = 0.3$  and  $v_0 = 0.4$ , (c)  $R = 0.6$  and  $v_0 = 0.4$ . All the data points above are obtained by averaging over 300 different realizations. The synchronization is much larger when  $\theta_R$  is small. Note that while the synchronization is significantly improved in the RASPO model, the convergence time increases when  $\theta_R$  is smaller.

decreasing the inherent noise. We also find that when  $\theta_R$  is large, a system of large  $n$  exhibits low synchronization, but when  $\theta_R$  is small, a system of large  $n$  exhibits high synchronization. These results suggest that when  $n$  approaches infinity, the synchronization shows a phase transition as a function of  $\theta_R$ . Thus, there seems to exist a critical restricted angle  $\theta_{Rc}$ , below which the synchronization is high, and above which the synchronization is low. The simulation results also indicate that  $\theta_{Rc}$  increases with  $v_0$  and decreases with  $R$ . Thus, it seems that our realistic angle restriction assumption significantly improves the synchronization of the SPO and thus also makes our model more practical for technological applications.

To study the synchronization  $V_\alpha$  as a function of the absolute velocity  $v_0$ , we perform numerical simulations of the RASPO

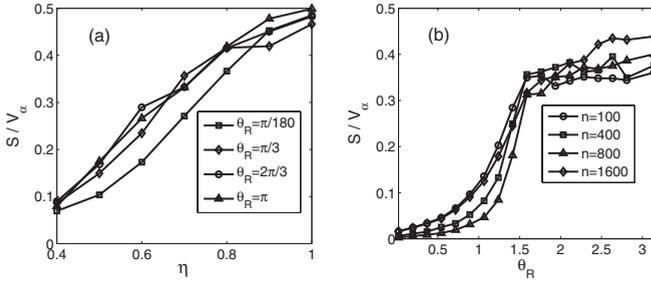


FIG. 3. (a) The relative standard deviation  $S/V_\alpha$  as a function of  $\eta$  for different  $\theta_R$ , where  $S$  is defined as  $S = [\frac{1}{N-1} \sum_{i=1}^N (V_\alpha - \bar{V}_\alpha)^2]^{1/2}$ . In the simulation, the density  $\rho = 1$ ,  $n = 200$ ,  $R = 0.6$ , and  $v_0 = 0.1$ . (b) The relative standard deviation of  $S/V_\alpha$  as a function of  $\theta_R$  for different  $n$ . All data points above are obtained by averaging over 400 different realizations.

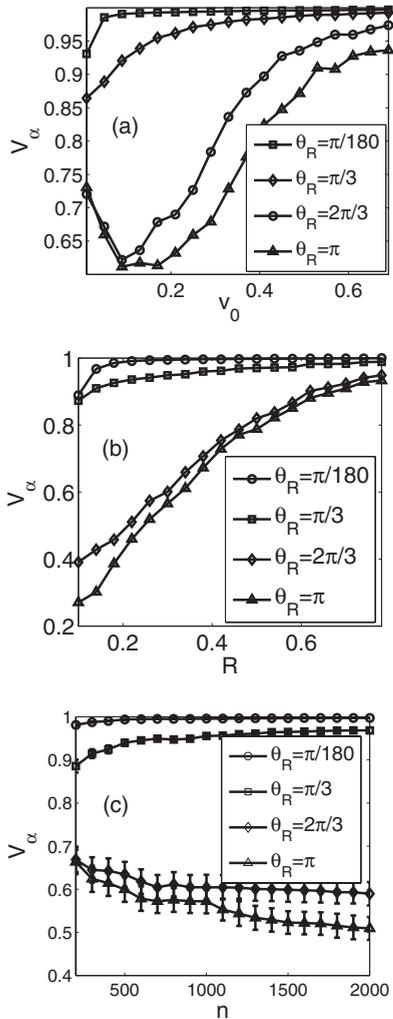


FIG. 4. (a) The synchronization  $V_\alpha$  as a function of the absolute velocity  $v_0$  for different restricted angle  $\theta_R$ . In the simulation,  $R = 0.3$ ,  $\rho = 1$ , and  $n = 800$ . (b) The synchronization  $V_\alpha$  as a function of radius  $R$  for different restricted angle  $\theta_R$ . In the simulation,  $v_0 = 0.1$ ,  $\rho = 1$ , and  $n = 800$ . (c) The synchronization  $V_\alpha$  as a function of system size  $n$  for different angle restricted angle  $\theta_R$ . In the simulation,  $R = 0.3$ ,  $\rho = 1$ , and  $v_0 = 0.1$ . All quantities are averaged over 300 realizations.

model for  $n = 400$ ,  $L = 20$ , and  $R = 0.3$  for various restricted angles  $\theta_R$  [Fig. 4(a)]. The simulation results demonstrate that the synchronization  $V_\alpha$  increases monotonically as the absolute velocity  $v_0$  increases when the restricted angle  $\theta_R$  is small, but  $V_\alpha$  decreases and increases (has a minimum) as the absolute velocity  $v_0$  increases when the restricted angle  $\theta_R$  is large. Thus, it is implicit that the synchronization is significantly improved by a small restricted angle, in contrast to the original VM for a wide range of absolute velocity  $v_0$ .

We also study the synchronization  $V_\alpha$  as a function of radius  $R$ . We perform the numerical simulations of RASPO for  $n = 400$ ,  $v_0 = 0.1$ , and various restricted angles  $\theta_R$  [see Fig. 4(b)]. Our simulation results indicate that, for a fixed  $\theta_R$ ,  $V_\alpha$  is an increasing function of  $R$ , which implies that the synchronization is significantly improved when the restricted angle is small. In particular, the synchronization is surprisingly improved more for worse conditions, i.e., when  $R$  is small.

The synchronization  $V_\alpha$  as a function of system size  $n$  for different restricted angles  $\theta_R$  is shown in Fig. 4(c). It is particularly surprising that the synchronization  $V_\alpha$  increases with  $n$  when the restricted angle  $\theta_R$  is small, which is in sharp contrast with being a decreasing function of  $n$  when the restricted angle  $\theta_R$  is large. Our simulation results also suggest that the synchronization converges to constant finite values when  $n$  is large.

We next study the effect of noise on restricted angle synchronization. In Fig. 5(a), we show the synchronization  $V_\alpha$  as a function of noise amplitude  $\eta$  for different restricted angles  $\theta_R$ . We see that  $V_\alpha$  is a decreasing function of  $\eta$  and, as  $\theta_R$  increases,  $V_\alpha$  decreases slowly and then quickly, indicating

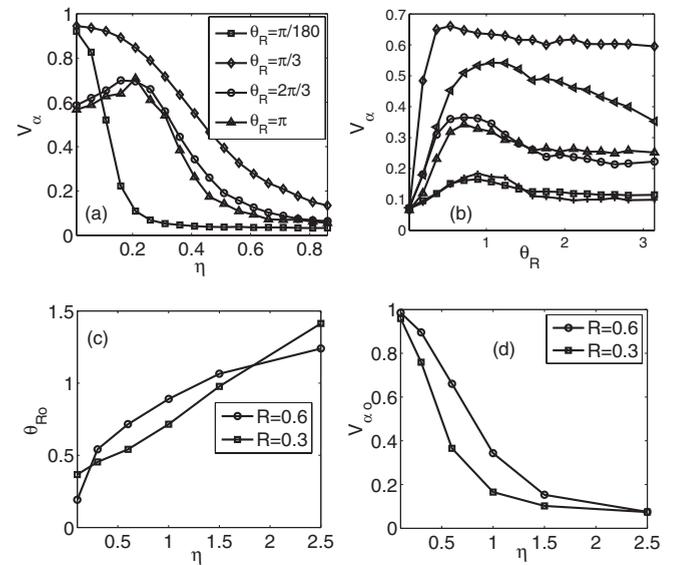


FIG. 5. (a) Synchronization  $V_\alpha$  as a function of noise amplitude  $\eta$  for different restricted angle  $\theta_R$ ,  $R = 0.6$ ,  $v_0 = 0.1$ . (b) Synchronization  $V_\alpha$  as a function of restricted angle  $\theta_R$  for  $\circ$ :  $\eta = 0.6, R = 0.3, v_0 = 0.1$ ;  $\square$ :  $\eta = 1, R = 0.3, v_0 = 0.1$ ;  $\diamond$ :  $\eta = 0.6, R = 0.6, v_0 = 0.1$ ;  $\triangle$ :  $\eta = 1, R = 0.6, v_0 = 0.1$ ;  $\triangleleft$ :  $\eta = 0.6, R = 0.3, v_0 = 0.3$ ,  $\blacktriangle$ :  $\eta = 1, R = 0.3, v_0 = 0.3$ . (c) Optimal restricted angle  $\theta_{Ro}$  as a function of noise amplitude  $\eta$ , for different  $R$  values.  $v_0 = 0.1$  (d) Optimal synchronization  $V_{\alpha o}$  as a function of  $\eta$  for different  $R$ . In the simulation,  $n = 200$  and all quantities are averaged over 300 realizations.

the existence of a peak when we plot the synchronization  $V_{\alpha}$  as a function of the restricted angle  $\theta_R$  [see Fig. 5(b)]. This means that, for each noise amplitude  $\eta$ , we can obtain an optimal value  $\theta_{Ro}$  for  $\theta_R$  [see Fig. 5(c)] and a corresponding optimal synchronization  $V_{\alpha o}$  [see Fig. 5(d)].

#### IV. CONCLUSION

In summary, we have proposed a restricted angle model that significantly (i) improves the synchronization of SPO systems when the restricted angle decreases because reducing  $\theta_R$  is effectively like reducing the internal noise which leads to improving synchronization, (ii) demonstrates the existence of a critical restricted angle  $\theta_{Rc}$  above which the synchronization order parameter changes sharply from a large value to a small value, and (iii) reveals that for each noise amplitude  $\eta$  the synchronization shows a peak as a function of  $\theta_R$ , so there exists an optimal  $\theta_{Ro}$  for which one will obtain the best synchronization  $V_{\alpha o}$ . Note that a model of SPO with restricted vision was studied, where an optimal view angle was found to obtain the fastest convergence speed [36]. However, this model is very different from the RASPO model since the

average direction of the objects within view angle can be in any direction and the angle change will be of all sizes. In contrast, in the RASPO model, the change of the angle is restricted.

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#### APPENDIX

In order to describe the RASPP model mathematically, we define the angle change function  $C(\alpha, \beta)$ , *Definition 1.* The angle change  $C(\alpha, \beta)$  denotes the change from angle  $\alpha$  to angle  $\beta$ , where  $\alpha, \beta \in [0, 2\pi)$ . So it is easy to obtain  $C(\alpha, \beta) = C(\beta, \alpha)$ .

The direction of the RASPP is updated as:

$$\theta_i(t+1) = \begin{cases} |\bar{\theta}_i(t) + \theta_R|_{2\pi} & \text{if } C(\theta_i(t), \bar{\theta}_i(t)) > \theta_R, \text{ and } C_l < C_r \\ \bar{\theta}_i(t) & \text{if } C(\theta_i(t), \bar{\theta}_i(t)) \leq \theta_R \\ |\bar{\theta}_i(t) - \theta_R|_{2\pi} & \text{if } C(\theta_i(t), \bar{\theta}_i(t)) > \theta_R, \text{ and } C_l > C_r \end{cases}$$

where  $\bar{\theta}_i(t)$  is obtained from Eq. (2),  $C_l = C([\bar{\theta}_i(t) + \theta_R]_{2\pi}, \bar{\theta}_i(t))$ ,  $C_r = C([\bar{\theta}_i(t) - \theta_R]_{2\pi}, \bar{\theta}_i(t))$ , and  $|\bullet|_{2\pi} = \bullet + 2k\pi \in [0, 2\pi)$ ,  $k \in \mathbb{Z}$ .

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