Cascade of failures in coupled network systems with multiple support-dependence relations

Jia Shao,[1] Sergey V. Buldyrev,[2,1] Shlomo Havlin,[3] and H. Eugene Stanley[1]
[1]*Center for Polymer Studies and Department of Physics, Boston University, Boston, Massachusetts 02215, USA*
[2]*Department of Physics, Yeshiva University, 500 West 185th Street, New York, New York 10033, USA*
[3]*Department of Physics, Bar-Ilan University, 52900 Ramat-Gan, Israel*
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We study, both analytically and numerically, the cascade of failures in two coupled network systems A and B, where multiple support-dependence relations are randomly built between nodes of networks A and B. In our model we assume that each node in one network can function only if it has at least a single support link connecting it to a functional node in the other network. We assume that networks A and B have (i) sizes N^A and N^B, (ii) degree distributions of connectivity links $P^A(k)$ and $P^B(k)$, (iii) degree distributions of support links $\tilde{P}^A(k)$ and $\tilde{P}^B(k)$, and (iv) random attack removes $(1-R^A)N^A$ and $(1-R^B)N^B$ nodes form the networks A and B, respectively. We find the fractions of nodes μ_∞^A and μ_∞^B which remain functional (giant component) at the end of the cascade process in networks A and B in terms of the generating functions of the degree distributions of their connectivity and support links. In a special case of Erdős-Rényi networks with average degrees a and b in networks A and B, respectively, and Poisson distributions of support links with average degrees $\tilde{a}$ and $\tilde{b}$ in networks A and B, respectively, $\mu_\infty^A = R^A[1-\exp(-\tilde{a}\mu_\infty^B)][1-\exp(-a\mu_\infty^A)]$ and $\mu_\infty^B = R^B[1-\exp(-\tilde{b}\mu_\infty^A)][1-\exp(-b\mu_\infty^B)]$. In the limit of $\tilde{a}\to\infty$ and $\tilde{b}\to\infty$, both networks become independent, and our model becomes equivalent to a random attack on a single Erdős-Rényi network. We also test our theory on two coupled scale-free networks, and find good agreement with the simulations.

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I. INTRODUCTION

In recent years, there has been extensive effort to study and understand the properties of complex networks. Previous research has mainly focused on properties of single networks which do not interact or depend on other networks [1–14]. Recently, the robustness of two interdependent coupled networks has been studied [15,16]. In interdependent networks, the failures of nodes in one network, A, will cause failures of dependent nodes in the other network, B, and vice versa. This process occurs recursively, and leads to a cascade of failures. It has been shown both analytically and numerically that the robustness of two interdependent networks is significantly lower compared to that of a single network [15]. Furthermore, the percolation transition in coupled networks is the first-order transition compared to the known second-order transition in a single network [15,16].

Previous studies on two interdependent coupled networks are restricted by the condition, that each node in network A depends on one and only one node in network B and vice versa [15]. However, in the real world, this assumption may not be valid. A single node in network A may depend on more than one node in network B and will function as long as at least one of its support nodes in network B is still functional. Similarly, a node in network B may depend on more than one support nodes in network A. As long as at least one of its support nodes functions, the node in network B will also function.

Examples of such systems include the coupled power grid network and the communication network which controls the power grid, where both networks depend on each other. In general, one power station provides power to more than one communication stations, and one communication station controls more than one power stations. As long as a communication station can obtain power from one power station, it can function properly. On the other hand, one communication station is sufficient to make one power station to function properly. However without any power, the communication station will fail, and without control the power station will stop working. Indeed in the 2003, due to failure of some power stations in Italy, the communication control system was damaged. This damage caused further fragmentation of the power grid, which finally led to a blackout in a significant part of Italy [17].

Under random attack, which is characterized by random removal of nodes in one or both networks, the coupled networks system demonstrate significantly different behavior from that of a single network [15]. The failures of nodes in network A can lead to the failures of dependent nodes in network B, and the failures of nodes in network B can produce a feedback on network A leading to further failures in network A. This process can occur recursively and can lead to a cascade of failures.

We provide a theoretical framework for understanding the robustness of interdependent networks with a random number of support and dependence relationships. Our theory agrees well with the numerical simulations of several model network systems, including coupled Erdős-Rényi (ER) [18] and coupled scale-free (SF) [2] networks. Our work extends previous works on coupled networks [15,16] from *one-to-one* support-dependence relation to *multiple* support-dependence relation. Our model can help to further understand real-life coupled network systems, where complex dependence-support relations may exist.

We define the stable state to be the state when the cascade of failures ends. We show that for two coupled ER networks the giant components of both networks in the stable state follow a simple law, which in the limit of a large number of support links is equivalent to random percolation of a single network. Our

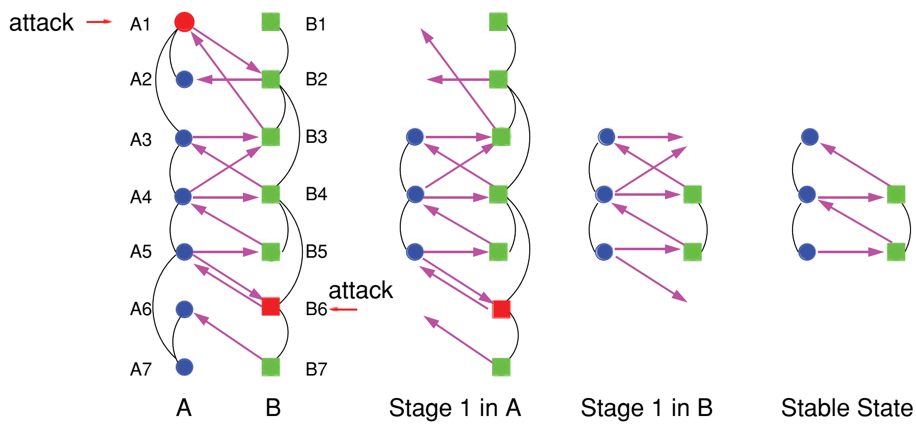

FIG. 1. (Color online) Demonstration of the stages of the cascade of failures in coupled networks A and B of size $N^A = N^B = 7$. Curves represent connectivity links within the network, while arrows (directed links) represent the support links connecting a support node in one network to the dependent node in the other network. Among the total 12 directed links, half of them $\tilde{a}N^A = 6$ represent the support from nodes in network B to nodes in network A (arrows from nodes in network B to nodes in network A). The rest $\tilde{b}N^B = 6$ links represent the support from nodes in network A to nodes in network B. The support-dependence relations between nodes in network A and network B are random. Initially, the attack is on node A_1 (shown in red) in network A and node B_6 (shown in red) in network B. The failed nodes are removed from the plot. In the first stage of the cascade of failures in network A, A_1 fails because of removal, node A_7 fails because of it has no support links, A_2 and A_6 fail because of separation from the giant component of network A. All the failed nodes will lead to failures of support links starting from them. Since we assume that the attack on network A occurs before that on network B, the support link from B_6 to A_5 is considered to be functional. In the first stage of the cascade of failures in network B, we first remove support links connecting network B to non-giant-component nodes in network A (B_3 to A_1, B_2 to A_2, B_7 to A_6). Next, node B_6 fails because of the attack and nodes B_1, B_2 and B_7 also fail because of no support. B_3 fails because it becomes separated from the giant component (nodes B_4 and B_5) of network B. Finally, after removing support links connecting nodes in network A to non-giant-component nodes in network B (A_3 to B_3, A_4 to B_3 and A_5 to B_6), the coupled network system reaches a stable state after one step in the cascade of failures, since all nodes in both giant components are connected and each node have at least one support node from the other network.

theory is relevant to a broad class of real-world interdependent network systems.

The paper is organized as follows. In Sec. II, we explain the model of the cascade of failures with multiple random support-dependence relations. In Sec. III, we derive analytically the process of the failure cascade. In Sec. IV, we present numerical tests on coupled ER and SF networks.

II. THE MODEL

We assume two networks A and B of sizes N^A and N^B and with given degree distributions, $P^A(k)$ and $P^B(k)$, of connectivity links connecting nodes in the same network (Fig. 1). The dependency relation is represented by a link connecting the support node in one network and the dependent node in the other network (support links). The support links between network A and network B are random and unidirectional. Initially (stage $n = 0$ of the failure cascade), each node in network A is supported by $\tilde{k}_A$ nodes in network B. We call $\tilde{k}_A$ a support degree of a node in network A. We connect a node in network A to its supporting nodes in network B by $\tilde{k}_A$ unidirectional support links represented by the arrows pointing from the support nodes in network B to this node in network A. We assume that $\tilde{k}_A$ is a random number taken from the support degree distribution $\tilde{P}^A(\tilde{k}_A)$. Similarly, the degree distribution of the support links feeding the nodes in network B by nodes in network A is $\tilde{P}^B(\tilde{k}_B)$. The support-dependence relations are random, i.e., for each node in network A its support nodes in network B are chosen at random and vice versa.

We assume that in order to be functional a node in network A must (i) have at least one functional support node in network B and (ii) must belong to the giant component of functional nodes in network A. Similarly, we assume that in order to be functional a node in network B must (i) have at least one functional support node in network A and (ii) must belong to the giant component of functional nodes in network B. The mathematical problem of finding functional nodes can be represented as a physical process of cascading failures in which the number of currently functional nodes decreases with time and eventually converges to the number of permanently functional nodes which satisfy conditions (i) and (ii).

The attack on the coupled network system is represented by a random removal of a fraction $1 - R^A$ of nodes in network A and a random removal of a fraction $1 - R^B$ of nodes in network B, where in general, $R^A \neq R^B$. The cascade of failures is demonstrated in Fig. 1. For small networks with $N^A = N^B = 7$, we show the case of random removal of one node in network A ($R^A = 6/7$) and one node in network B ($R^B = 6/7$). At each stage of the cascade of failures, both networks experience further failures and the number of functional nodes decreases with time. Without loss of generality, we assume the random attack on network A occurs before that on network B. Thus, when we analyze the first stage of the cascade of failures in network A, all the nodes in network B are considered functional. At each stage, the nodes which do not have any currently functional support nodes from the other network, and the nodes which are separated from the giant component of currently functional nodes in their network, are

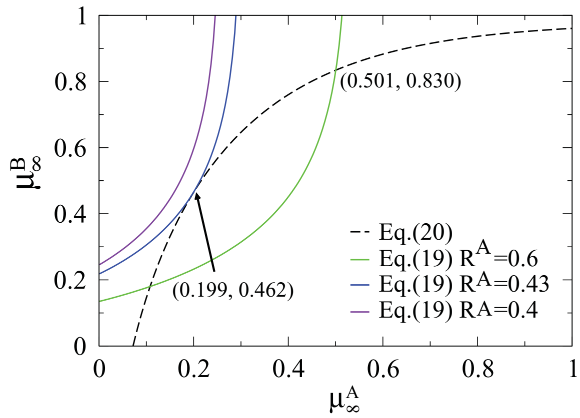

FIG. 2. (Color online) Demonstration of the functional relation between μ_∞^B and μ_∞^A in Eqs. (19) and (20) for a system of two coupled ER networks with $a = b = 4$, $\tilde{b} = \tilde{a} = 4$, and $R^B = 1$ at different values of R^A. Since we use $R^B = 1$, at different R^A, the relation between μ_∞^B and μ_∞^A given by Eq. (20) remains the same (shown by the dashed line). Equation (19) with $R^A = 0.6, 0.43,$ and 0.4 is shown. One can see that when $R^A < 0.43$, only a trivial solution ($\mu_\infty^A = 0, \mu_\infty^B = 0$) exists for Eqs. (19) and (20). The value $1 - R_c^A \equiv 1 - 0.43 = 0.57$ represents the maximum fraction of nodes in network A one can randomly remove at the initial stage of the cascade of failures for which the non-zero giant components of both networks still exist at the stable state. The abrupt disappearance of the nontrivial solution at $R^A < R_c^A$ corresponding to complete fragmentation of both networks represents the first order nature of the percolation phase transition for coupled networks.

considered to have failed. This process will continue until no further node failure in either network occurs. At this stage the currently functional nodes in both networks satisfy both conditions (i) and (ii) of functional nodes. We will call the giant components of functional nodes the mutually supported giant components.

III. ANALYTICAL SOLUTION

The stable state of the two mutually supported giant components in both networks is usually reached after several stages in the cascade of failures. In each stage of the cascade of failures, we analyze first network A then network B. Such a procedure does not have any effect on the final result of the failure cascade. While treating nodes in network A at stage n, we assume that all their support nodes in network B which are found to be functional at the previous $(n-1)$ stage are still functional. When treating nodes in network B at stage n, we assume that all their support nodes in network A which are found to be functional at the current (nth) stage are still functional. We denote the fractions of currently functional nodes at stage n in networks A and B as μ_n^A and μ_n^B, respectively. We assume that initially $\mu_0^B = 1$.

Since for each node in network A $\tilde{k}_A$, we randomly set up $\tilde{k}_A$ support nodes in network B, the probability that this node at stage n has no functional support nodes in network B is

$$r_n^A = \sum_{\tilde{k}_A=0}^{\infty} \tilde{P}^A(\tilde{k}_A)\left(1-\mu_{n-1}^B\right)^{\tilde{k}_A} = \tilde{G}^A\left(1-\mu_{n-1}^B\right), \quad (1)$$

where $\tilde{G}^A$ is the generating function of the support degree distribution $\tilde{P}^A(\tilde{k}_A)$. Analogously, the probability that a node in network B at stage n has no functional support nodes in network A is

$$r_n^B = \sum_{\tilde{k}_B=0}^{\infty} \tilde{P}^B(\tilde{k}_B)\left(1-\mu_n^A\right)^{\tilde{k}_B} = \tilde{G}^B\left(1-\mu_n^A\right). \quad (2)$$

.

We can easily generalize Eqs. (1) and (2) to the case when a fraction of nodes $1-q^A$ in network A is totally independent of nodes in network B and fraction of nodes $1-q^B$ in network B is totally independent of nodes in network A [16], by formally assigning to these autonomous nodes infinite number of support nodes in the other network, meaning that even total failure of the other network will not destroy them. In this case $\tilde{P}^A(\infty) = 1-q^A$ and $\tilde{P}^B(\infty) = 1-q^B$. Thus we must simply renormalize generating functions of the support degree distributions:

$$\begin{cases} \tilde{G}^A(x) \equiv q^A \sum_{k=0}^{\infty} \tilde{P}^A(k)x^k, \\ \tilde{G}^B(x) \equiv q^B \sum_{k=0}^{\infty} \tilde{P}^B(k)x^k, \end{cases} \quad (3)$$

In addition to nonautonomous nodes with no functional support nodes in the other network, some nodes become nonfunctional due to initial attack. Accordingly, the fractions of nodes in network A which remain functional at stage n after application of condition (i) is

$$p_n^A = R^A\left(1-r_n^A\right). \quad (4)$$

Analogously, the fraction of nodes in network B which do not fail at stage n due to condition (i) is

$$p_n^B = R^B\left(1-r_n^B\right). \quad (5)$$

Now we will find the subset of nodes that remain functional after stage n. We will present our analysis only for network A because a completely analogous analysis is valid for network B. The networks A and B are connected randomly. Thus, from the point of view of network A the fraction of nodes $1-p_n^A$ that become nonfunctional due to application of condition (i) are randomly removed. We will denote this decimated network A_n and its giant component $\mathcal{G}A_n$. We will show that $\mathcal{G}A_n$ coincides with the set of nodes $\mathcal{F}A_n$ which remain functional after stage n by the method of mathematical induction. At the first stage $\mathcal{G}A_1 = \mathcal{F}A_1$ because originally all the nodes of network A are considered functional. On later stages this is not obvious because some nodes of A_n have become nonfunctional at the previous stage since they do not belong to the giant component of the currently functional nodes. We will assume that $\mathcal{G}A_{n-1} = \mathcal{F}A_{n-1}$ and show that $\mathcal{G}A_n = \mathcal{F}A_n$. It is obvious that $\mathcal{F}A_n \subseteq \mathcal{G}A_n$, because only nodes which belong to $\mathcal{G}A_n$ can be functional. We also note that $A_n \subseteq A_{n-1}$ and hence in thermodynamic limit $\mathcal{G}A_n \subseteq \mathcal{G}A_{n-1}$. But due to the induction assumption $\mathcal{G}A_{n-1} = \mathcal{F}A_{n-1}$. Thus all the nodes of $\mathcal{G}A_n$ are currently functional because they pass both criteria: they belong to the giant component of the nodes supported by the currently functional nodes in network B and all of them remained functional after the previous stage. Thus indeed $\mathcal{F}A_n = \mathcal{G}A_n$. The same is true for the network B: $\mathcal{F}B_n = \mathcal{G}B_n$

Thus, the fractions of functional nodes after stage n, μ_n^A and μ_n^B are equal to the fraction of nodes in the giant components of networks A and B, respectively, after random removal of fractions $1 - p_n^A$ and $1 - p_n^B$, respectively. Since networks A and B are randomly connected, we can use the apparatus of generating functions [21] to derive the analytical forms of μ_n^A and μ_n^B.

The generating functions of the degree distribution $P^A(k)$ of network A and $P^B(k)$ of network B are

$$\begin{cases} G_{A0}(x) \equiv \Sigma_{k=0}^{\infty} P^A(k)x^k, \\ G_{B0}(x) \equiv \Sigma_{k=0}^{\infty} P^B(k)x^k. \end{cases} \quad (6)$$

Analogously, the generating functions of the underlying branching processes are

$$\begin{cases} G_{A1}(x) \equiv G'_{A0}(x)/G'_{A0}(1), \\ G_{B1}(x) \equiv G'_{B0}(x)/G'_{B0}(1). \end{cases} \quad (7)$$

After random removal of a fraction $1 - p$ of nodes, the remaining fraction p of the network will have different degree distribution. The new generation functions G_0 and G_1 will be [19,20]

$$\begin{cases} G_{A0}(x,p) = G_{A0}(1 - p(1 - x)), \\ G_{B0}(x,p) = G_{B0}(1 - p(1 - x)), \\ G_{A1}(x,p) \equiv G_{A1}(1 - p(1 - x)), \\ G_{B1}(x,p) \equiv G_{B1}(1 - p(1 - x)). \end{cases} \quad (8)$$

According to the results on single networks [19,20], after random removal of a fraction $1 - p_n^A$ (or $1 - p_n^B$) of nodes of the original networks A and B, the fractions of nodes that belong to the giant components of the remaining network A_n or network B_n, which have p_n^A or p_n^B fractions of nodes of the original networks A and B, respectively, are

$$\begin{cases} g^A(p_n^A) = 1 - G_{A0}(f_n^A, p_n^A), \\ g^B(p_n^B) = 1 - G_{B0}(f_n^B, p_n^B), \end{cases} \quad (9)$$

where f_n^A and f_n^B satisfy transcendental equations

$$\begin{cases} f_n^A = G_{A1}(f_n^A, p_n^A), \\ f_n^B = G_{B1}(f_n^B, p_n^B). \end{cases} \quad (10)$$

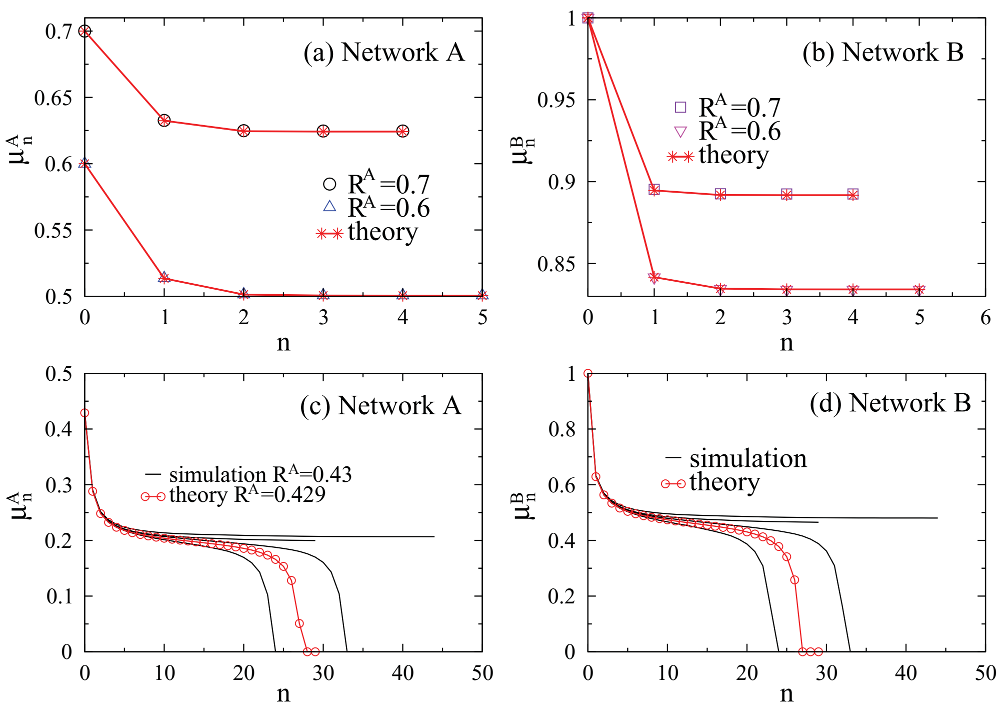

FIG. 3. (Color online) The case of coupled ER networks. Comparison between the theoretical predictions, obtained from Eqs. (12), (15), (11), and (18), and numerical simulations with $N^A = N^B = 10^6$, $a = b = 4$, $\tilde{b} = \tilde{a} = 4$, $R^B = 1$, and several values of R^A. (a) and (b) show μ_n^A and μ_n^B at different stages n of the cascade of failures for $R^A = 0.7$ and $R^A = 0.6 > R_c^A \approx 0.43$ for both theory (lines) and simulations (symbols). One can see that both μ_n^A and μ_n^B approach a stable value μ_∞^A and μ_∞^B at the end of the cascade of failures. The agreement between theory and numerical simulations is very good. (c) and (d) show μ_n^A and μ_n^B at different stages n of the cascade of failures for $R^A \approx R_c^A$. The bare lines represent several realizations of the simulations and the lines with symbols represent the theoretical predictions. One can see that for the early stages (small n) the agreement is good, however at large n the deviation due to random fluctuations in the actual fraction of the giant component starts to increase. The random realizations split into two classes: one that converges to a non-zero giant component for both networks and the other that results in a complete fragmentation.

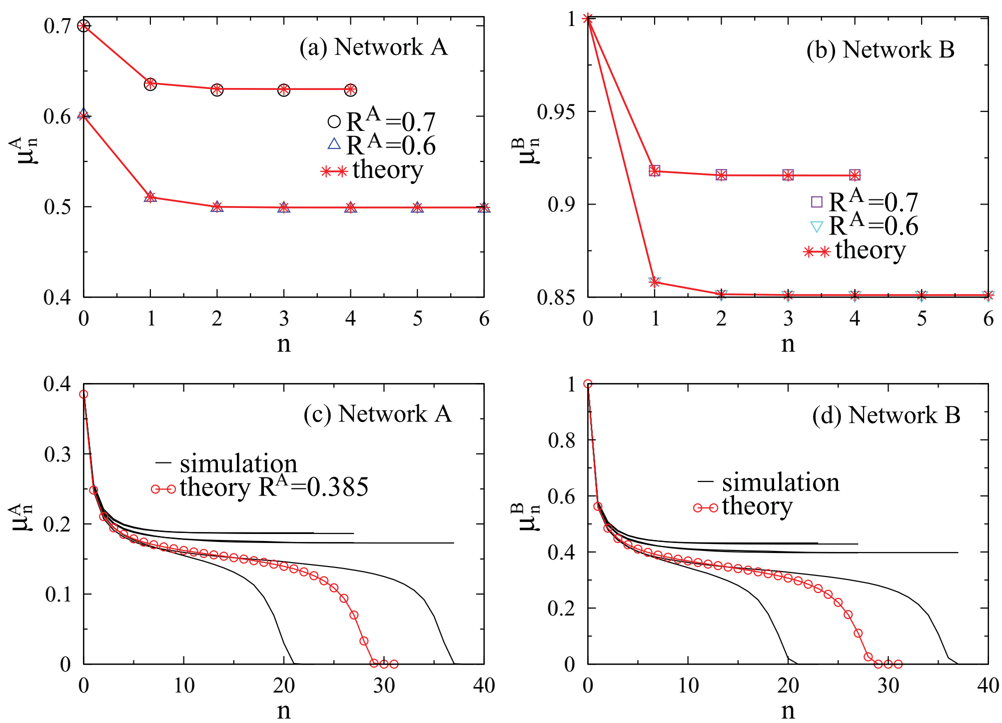

FIG. 4. (Color online) The case of coupled SF networks. Comparison between the theoretical predictions, obtained from Eqs. (12), (9), (10), and (18), and numerical simulations with $N^A = N^B = 10^6$, $\lambda^A = \lambda^B = 2.5$, $\tilde{b} = \tilde{a} = 4$, $R^B = 1$, and different values of R^A. (a) and (b) show μ_n^A and μ_n^B at different stages n of the cascade of failures for $R^A = 0.7$ and $R^A = 0.6 > R_c^A \approx 0.385$ for both theory (lines) and simulations (symbols). Similar to Fig. 3, one can see that both μ_n^A and μ_n^B approach a stable value μ_∞^A and μ_∞^B at the end of cascade failures. The agreement between the theory and numerical simulations is very good. (c) and (d) show μ_n^A and μ_n^B at different stages n of the cascade of failures for $R^A \approx R_c^A$. Bare lines represent several realizations of the simulations and the lines with symbols represent the theoretical predictions. One can see that for the early stages the agreement is good, however at large n the deviation due to random fluctuations in the actual fraction of the giant component increase. The random realizations split into two classes: one that converges to a nonzero giant component for both networks and the other that results in a complete fragmentation. For coupled SF networks, at $R^A = R_c^A$, the fluctuations of both μ_n^A and μ_n^B seem to be relatively larger than that of coupled ER networks, due to the existence of large degree nodes in SF networks.

Thus μ_n^A and μ_n^B, the fractions of nodes in the giant components relative to the original sizes of network A and network B [19,20] are

$$\begin{cases} \mu_n^A = p_n^A g^A(p_n^A), \\ \mu_n^B = p_n^B g^B(p_n^B). \end{cases} \tag{11}$$

Combining Eq. (11) and Eqs. (1), (2), (4), and (5), we can express the cascade of failures in networks A and B in the thermodynamic limit $N^A \to \infty$, $N^B \to \infty$ in terms of iterations:

$$\begin{cases} \mu_0^B = 1, \\ p_1^A = R^A\left[1 - \tilde{G}^A\left(1 - \mu_0^B\right)\right], \\ \mu_1^A = p_1^A g^A\left(p_1^A\right), \\ \ldots, \\ p_n^A = R^A\left[1 - \tilde{G}^A\left(1 - \mu_{n-1}^B\right)\right], \\ \mu_n^A = p_n^A g^A\left(p_n^A\right), \\ p_n^B = R^B\left[1 - \tilde{G}^B\left(1 - \mu_n^A\right)\right], \\ \mu_n^B = p_n^B g^B\left(p_n^B\right). \end{cases} \tag{12}$$

When the cascade of failures process stops, f_n^A, f_n^B, p_n^A, p_n^B, μ_n^A, and μ_n^B all reach the constant values, f_∞^A, f_∞^B, p_∞^A, p_∞^B, μ_∞^A, and μ_∞^B, respectively. These final values can be readily found from the set of equations:

$$\begin{cases} f_\infty^A = G_{A1}(1 - p_\infty^A(1 - f_\infty^A)), \\ f_\infty^B = G_{B1}(1 - p_\infty^B(1 - f_\infty^B)), \\ p_\infty^A = R^A[1 - \tilde{G}^A(1 - \mu_\infty^B)], \\ p_\infty^B = R^B[1 - \tilde{G}^B(1 - \mu_\infty^A)], \\ \mu_\infty^A = p_\infty^A[1 - G_{A0}(1 - p_\infty^A(1 - f_\infty^A))], \\ \mu_\infty^B = p_\infty^B[1 - G_{B0}(1 - p_\infty^B(1 - f_\infty^B))]. \end{cases} \tag{13}$$

The functional forms of $\tilde{G}^A$, $\tilde{G}^B$, G_{A1}, G_{B1}, G_{A0}, and G_{B0} can be complicated, thus Eqs. (13) can be solved only numerically for most cases, including coupled SF networks. However, for ER networks, $G_0(x)$ and $G_1(x)$ have the same simple form [21]

$$G_0(x) = G_1(x) = e^{\langle k\rangle(x-1)}, \tag{14}$$

where for network A, $\langle k\rangle = a$ and for network B, $\langle k\rangle = b$. Thus, the above process of the cascade of failures can be significantly simplified. Equations (9) can be reduced to

$$\begin{cases} g^A(p_\infty^A) = 1 - f_\infty^A = 1 - e^{ap_\infty^A(f_\infty^A - 1)} = 1 - e^{-a\mu_\infty^A}, \\ g^B(p_\infty^B) = 1 - f_\infty^B = 1 - e^{bp_\infty^B(f_\infty^B - 1)} = 1 - e^{-b\mu_\infty^B}. \end{cases} \tag{15}$$

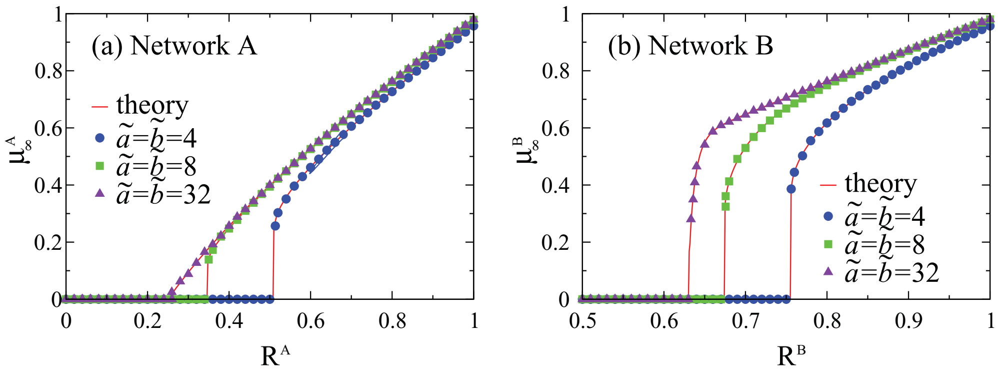

FIG. 5. (Color online) The fraction of the giant components of network (a) A, μ_∞^A and network (b) B, μ_∞^B in the stable states as a function of R^A and R^B for coupled ER networks A and B with $N^A = N^B = 10^6$, $a = b = 4$, $\tilde{b} = \tilde{a} = 4$, and $1 - R^A = 2(1 - R^B)$. Several curves for $\tilde{b} = \tilde{a} = 4$, $\tilde{b} = \tilde{a} = 8$, and $\tilde{b} = \tilde{a} = 32$ are shown. The theory (lines) agrees very well with the simulation results (symbols). One can see that for a given set of $\tilde{b}$ and $\tilde{a}$ there exist critical thresholds R_c^A and R_c^B, below which both networks will collapse and have no stable nonzero giant components. The value of R_c^A approaches the critical threshold of random percolation ($r = 1/a = 0.25$) of a single network for large values of $\tilde{b}$ and $\tilde{a}$. The initial attack on network B is smaller than that on network A and thus $R_c^B > R_c^A$.

Using Eq. (15) we can exclude p_∞^A, p_∞^B, f_∞^A, and f_∞^B from Eqs. (13). Thus for the stable state of two coupled ER networks, the mutually supported giant components satisfy a system of two equations:

$$\mu_\infty^A = R^A[1 - \tilde{G}^A(1 - \mu_\infty^B)]\left(1 - e^{-a\mu_\infty^A}\right), \tag{16}$$

$$\mu_\infty^B = R^B[1 - \tilde{G}^B(1 - \mu_\infty^A)]\left(1 - e^{-b\mu_\infty^B}\right). \tag{17}$$

Another simplification can be achieved if we assume that the support degree distributions $\tilde{P}^A(\tilde{k})$ and $\tilde{P}^B(\tilde{k})$ are Poisson distributions with average degrees $\tilde{a}$ and $\tilde{b}$, respectively. We also assume that there are no autonomous nodes ($q^A = q^B = 1$). In this case

$$\begin{cases} \tilde{G}^A(1 - \mu_\infty^B) = e^{-\tilde{a}\mu_\infty^B}, \\ \tilde{G}^B(1 - \mu_\infty^A) = e^{-\tilde{b}\mu_\infty^A}, \end{cases} \tag{18}$$

Plugging Eq. (18) into Eqs. (16) and (17) we obtain remarkably symmetric percolation laws for coupled ER networks with Poisson distribution of supporting links:

$$\mu_\infty^A = R^A\left(1 - e^{-\tilde{a}\mu_\infty^B}\right)\left(1 - e^{-a\mu_\infty^A}\right), \tag{19}$$

$$\mu_\infty^B = R^B\left(1 - e^{-\tilde{b}\mu_\infty^A}\right)\left(1 - e^{-b\mu_\infty^B}\right). \tag{20}$$

Equations (19) and (20) are simple and can be related to the theory of random percolation of a single ER network [18,22,23], for which the fraction of the giant component is $\mu_\infty = R(1 - e^{-\langle k\rangle\mu_\infty})$. The coupled ER networks bring new terms $1 - e^{-\tilde{a}\mu_\infty^B}$ and $1 - e^{-\tilde{b}\mu_\infty^A}$. In the limit of $\tilde{b} \to \infty$ (or $\tilde{a} \to \infty$), the giant component of network B (or network A) does not depend on the other network and behaves similarly to the giant component in random percolation of a single network.

The system of Eqs. (19) and (20) can be easily solved graphically (Fig. 2). Equation (19) has a trivial solution $\mu_\infty^A = 0$. For $\mu_\infty^A > 0$ it defines a function $\mu_\infty^B(\mu_\infty^A)$ which is an elementary function of μ_∞^A. It is easy to show that this function is continuous, positive, and has positive first and second derivatives in the interval $[0,\mu_m^A)$, where $\mu_m^A < 1$ and μ_∞^B approaches $+\infty$ asymptotically at $\mu_\infty^A = \mu_m^A$. Analogous facts are valid for $\mu_\infty^A(\mu_\infty^B)$, which can be found from Eq. (20). This equation also has a trivial solution $\mu_\infty^B = 0$. Thus, the graphical solution is given either by intersection of straight lines $\mu_\infty^B = 0$ and $\mu_\infty^A = 0$ or by intersection of the curves $\mu_\infty^A(\mu_\infty^B)$ and $\mu_\infty^B(\mu_\infty^A)$ which due to their convexity my have at most two intersections at positive $\mu_\infty^B > 0$ and $\mu_\infty^A > 0$. The physical solution corresponds to the largest solution, because the iterative process (12) starts from $\mu_0^B = 1$ and can converge only to the largest of these two solutions.

From Eqs. (19) and (20), we find μ_∞^A and μ_∞^B for a given set of parameters a, b, $\tilde{a}$, $\tilde{b}$, R^A, and R^B. However, for some values of these parameters, positive solutions for μ_∞^A and μ_∞^B corresponding to mutually supported giant components may not exist. If we fix the values of five parameters (for example R^B,a,b,$\tilde{a}$ and $\tilde{b}$) we can find a critical threshold for the sixth parameter (for example R^A) above which the two coupled ER networks have non-zero mutually supported giant components (see Fig. 2). We will denote these thresholds as $R_c^A(a,b,\tilde{a},\tilde{b},R^B)$, $R_c^B(a,b,\tilde{a},\tilde{b},R^B)$, $\tilde{a}_c(a,b,\tilde{b},R^A,R^B)$, $\tilde{b}_c(a,b,\tilde{a},R^A,R^B)$, $a_c(b,\tilde{a},\tilde{b},R^A,R^B)$, and $b_c(a,\tilde{a},\tilde{b},R^A,R^B)$, which form a five-dimensional critical manifold in the six-dimensional parameter space. The critical value of the parameters can be obtained by finding the tangent point of the two curves $\mu_\infty^B(\mu_\infty^A)$ and $\mu_\infty^A(\mu_\infty^B)$ found from Eqs. (19) and (20), respectively (Fig. 2). Thus, the critical manifold can be found from the tangential condition

$$\left.\frac{d\mu_\infty^A}{d\mu_\infty^B}\right|_{\text{Eq.(20)}} \left.\frac{d\mu_\infty^B}{d\mu_\infty^A}\right|_{\text{Eq.(19)}} = 1, \tag{21}$$

together with Eqs. (19) and (20). Once any of the parameters decreases by a infinitesimal quantity from the value it has at a point on the critical manifold the positive solution corresponding to the existence of mutually supported giant components disappears and only the trivial solution corresponding to complete disintegration of the networks is left. This is a typical behavior for the first order phase transition at which both order parameters, μ_∞^A and μ_∞^B, discontinuously change from positive values to zero. Only when $\tilde{a} \to \infty$ or

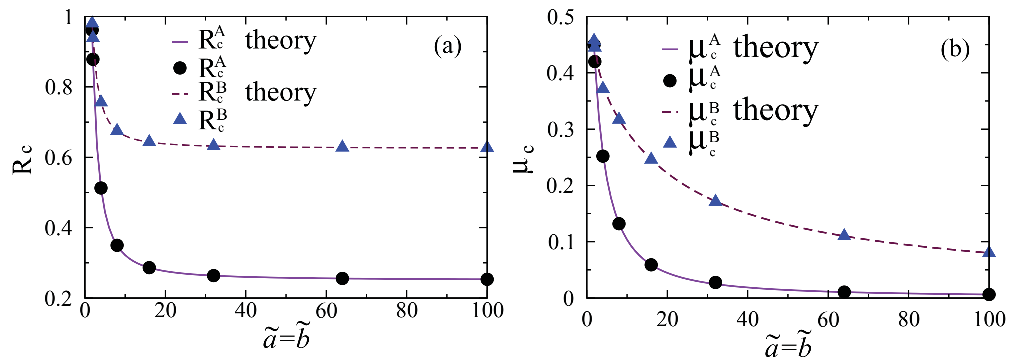

FIG. 6. (Color online) The dependences of (a) R_c and (b) μ_c on $\tilde{b} = \tilde{a}$ for coupled ER networks with $N^A = N^B = 10^6$, $a = b = 4$, and $1 - R^A = 2(1 - R^B)$. In (a), the critical initial fraction of the network A, R_c^A, and the critical initial fraction of the network B, R_c^B, are shown as a function of $\tilde{b} = \tilde{a}$. The theory (full line and dashed lines) fits well the simulation results (symbols). Note that R_c^A approaches the critical threshold $(1/a)$ of random percolation for a single network, as also predicted by Eqs. (19) and (20). In (b), the giant component of both network μ_c^A and μ_c^B are shown as a function of $\tilde{b} = \tilde{a}$ at R_c^A and R_c^B. The theory (full line and dashed line) fits well the simulation results (symbols). One can see that for large $\tilde{b} = \tilde{a}$, μ_c^A and μ_c^B both approach zero as expected for a single network. However, μ_c^A and μ_c^B will never reach zero for finite $\tilde{b}$ and $\tilde{a}$ and the phase transition thus remain a first order.

$\tilde{b} \to \infty$ the behavior of networks decouples and μ_∞^A or μ_∞^B can gradually approach zero with the decrease of one of the remaining parameters. Thus only if $\tilde{a} \to \infty$ or $\tilde{b} \to \infty$, the behavior of the mutually supported giant components can be described by the second-order phase transition, while in all other cases (in the absence of autonomous nodes) it is described by a first-order transition. In case when autonomous nodes are present, the order of the transition can change from first to second if the values of q_A or q_B are sufficiently small, because functions $\mu_\infty^B(\mu_\infty^A)$ and $\mu_\infty^A(\mu_\infty^B)$ may become negative at small values of their arguments. In this case, the analysis of Eqs. (19) and (20) is similar to the analysis in Ref. [16].

IV. NUMERICAL SIMULATIONS

Next, we compare our theoretical results obtained in Sec. III to results of numerical simulations. For simplicity, in this section we will study only the cases of the Poisson support degree distributions in both networks and $q_A = q_B = 1$. We begin with comparing the simulations of the stage n of the failure cascade in coupled ER networks with our theoretical predictions. In all our simulations, we use $N^A = N^B = 10^6$. Figure 3 shows μ_n^A and μ_n^B as a function of n for $a = b = 4$, $\tilde{b} = \tilde{a} = 4$, $R^B = 1$ and for different values of R^A. One sees very good agreement between the theory and the simulations. Close to R_c^A, both μ_n^A and μ_n^B show large fluctuations between different realizations (shown in Figs. 3(c) and 3(d)). The random realizations split into two classes: one that converges to a nonzero giant component for both networks and the other results in a complete fragmentation. The agreement between the simulations and theoretical predictions is also good for different values of R^B, a, b and $\tilde{b}$ and $\tilde{a}$.

In Fig. 4, we compare the theoretical predictions and simulations of the giant components at different stages of the cascade of failures for a system of two coupled SF networks with $\lambda^A = \lambda^B = 2.5$, $\tilde{b} = \tilde{a} = 4$, $R^B = 1$, and different values of R^A. Similarly, we obtain agreement between the theoretical predictions and the simulations. Close to R_c^A, both μ_n^A and μ_n^B of different random realizations show large fluctuations and different realizations also split into two classes. We also simulate other values of R^B, λ^A, λ^B, $\tilde{b}$, and $\tilde{a}$ and found very good agreement between and theoretical predictions and simulations.

The fractions of the giant components of both networks A (μ_∞^A) and B (μ_∞^A) for coupled ER networks, in the stable state can be derived from Eqs. (19) and (20). We solve these equations numerically for different values of R^A and R^B, and compare the theoretical predictions with the simulation results (Fig. 5). For simplicity, we assume $a = b = 4$ and that the initial fraction of nodes affected by the random attack in network A is twice as large as that in network B [$1 - R^A = 2(1 - R^B)$]. We test our theory for various average degrees of support links assuming that $\tilde{b} = \tilde{a}$.

In Fig. 5, we present results for the giant components of both networks as a function of R^A and R^B. We find that the theory agrees well with simulation results for different sets of $\tilde{b}$ and $\tilde{a}$. In Fig. 5, one can also clearly identify the critical R_c^A and R_c^B, which are the minimum fractions of both networks needed to be kept at the beginning of the cascade of failures in order to have nonzero connected giant components of both networks at the stable state. At R_c^A and R_c^B, both μ_∞^A and μ_∞^B show an abrupt change from a finite fraction (μ_c^A and μ_c^B) to zero. As $\tilde{b}$ and $\tilde{a}$ increase, R_c^A approaches the critical threshold of random percolation of a single ER network, which is $1/a = 1/4$. As expected for single networks, μ_c^A and μ_c^B approach 0 and a second-order phase transition emerges for infinite $\tilde{b}$ and $\tilde{a}$. However, for finite $\tilde{b}$ and $\tilde{a}$ the changes of μ_∞^A and μ_∞^B are not continuous at R_c^A and R_c^B, indicating a first-order phase transition. This result is predicted by Eqs. (19) and (20). We find that the theory agrees well with the simulation results for the entire range of R_c^A and R_c^B and for different values of $\tilde{b}$ and $\tilde{a}$.

Next, we study both theoretically and numerically the dependence of R_c^A and R_c^B, μ_c^A and μ_c^B on $\tilde{b}$ and $\tilde{a}$ (see Fig. 6). For simplicity and for comparing with our earlier cases, we use the same set of parameters for both networks: $a = b = 4$, $\tilde{b} = \tilde{a}$, and $1 - R^A = 2(1 - R^B)$. As seen from

Fig. 6, the theory agrees well with the numerical simulations. For large $\tilde{b} = \tilde{a}$, one can see that R_c^A approaches the random percolation threshold $1/a$ of a single ER network. This behavior indicates that when network A has enough support from network B or *vice versa*, both networks will behave as if they are independent. Indeed, for large $\tilde{b}$ and $\tilde{a}$, at R_c^A and R_c^B, the stable giant components of both networks μ_c^A and μ_c^A approach zero as expected for a second order percolation phase transition. However, as seen from Eqs. (19) and (20), for finite values of $\tilde{b}$ and $\tilde{a}$, neither μ_c^A nor μ_c^B is zero. This result supports the existence of a first-order phase transition for the entire range of $\tilde{b}$ and $\tilde{a}$. Good agreement between theory and simulations, and similar behavior of R_c^A, R_c^B, μ_c^A, and μ_c^A as a functions of $\tilde{b}$ and $\tilde{a}$ have been found for other sets of parameters.

V. CONCLUSIONS AND DISCUSSIONS

In this paper, we extend previous works [15,16] on the cascade of failures on interdependent networks by considering multiple support-dependence relations between two coupled network systems. Our theory is in excellent agreement with the numerical simulations on coupled Erdős-Rényi (ER) and coupled scale-free (SF) networks systems. For coupled ER networks, the percolation law for the mutually supported giant components of both networks have a simple analytical form, which in the limit of large number of supports gives the percolation law for single networks. Only in the limit of infinite number of support links or in case of the existence of autonomous nodes which do not need any support from the other network, the behavior of the mutually supported giant components can be described by the second-order percolation phase transition while in all other cases, the coupled networks in our model disintegrate via a first order phase transition. Our model can help to further understand real-life coupled network systems, where complex dependence-support relations exists. Recently, a complementary approach to study the robustness of coupled networks system has been proposed [24], which is based on a quite different assumption about the way networks are coupled. In contrast to our case where $p_c = R_c^A$ or $p_c = R_c^B$ increases due to coupling, in their case p_c decreases. Note that there are also recent efforts to study the robustness of single networks [25–27] undergoing targeted percolation, which correlates with the topology of the network. In the same spirit, our work can be extended to study the robustness of coupled networks under nonrandom percolation. A first attempt in this direction for interdependent networks can be found in Ref. [28]. The case of robustness of the general case of n coupled networks, network of networks, has been studied very recently [29].

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[1] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[2] A. L. Barabási and R. Albert, Science **286**, 509 (1999).
[3] R. Pastor-Satorras and A. Vespignani, *Evolution and Structure of the Internet: A Statistical Physics Approach* (Cambridge University Press, Cambridge, 2006).
[4] S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks: From Biological Nets to the Internet and WWW* (Oxford University Press, New York, 2003).
[5] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. **85**, 4626 (2000).
[6] L. A. Braunstein, Z. Wu, Y. Chen, S. V. Buldyrev, T. Kalisky, S. Sreenivasan, R. Cohen, E. Lopez, S. Havlin, and H. E. Stanley, International Journal of Bifurcation and Chaos **17**, 2215 (2007).
[7] J. Shao, S. Havlin, and H. E. Stanley, Phys. Rev. Lett. **103**, 018701 (2009).
[8] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E **64**, 026118 (2001).
[9] M. E. J. Newman, *Networks: An Introduction*, (Oxford University Press, New York, 2010).
[10] R. Cohen and S. Havlin, *Complex Networks: Structure, Robustness and Function*, (Cambridge University Press, Cambridge, 2010).
[11] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Phys. Rep. **424**, 175 (2006).
[12] G. Caldarelli and A. Vespignani, *Large Scale Structure and Dynamics of Complex Networks* (World Scientific, Singapore, 2007).
[13] A. Barrat, M. Barthélemy, and A. Vespignani, *Dynamical Processes on Complex Networks* (Cambridge University Press, Cambridge, 2008).
[14] D. S. Callaway, M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. Lett. **85**, 5468 (2000).
[15] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin, Nature (London) **464**, 1025 (2010).
[16] R. Parshani, S. V. Buldyrev, and S. Havlin, Phys. Rev. Lett. **105**, 048701 (2010).
[17] V. Rosato, L. Issacharoff, F. Tiriticco, S. Meloni, S. Porcellinis, and R. Setola, Int. J. Critical Infrastructures **4**, 63 (2008).
[18] P. Erdős and A. Rényi, Publ. Math. **6**, 290 (1959); Publ. Math. Inst. Hung. Acad. Sci. **5**, 17 (1960).
[19] J. Shao, S. V. Buldyrev, R. Cohen, M. Kitsak, S. Havlin, and H. E. Stanley, Europhys. Lett. **84**, 48004 (2008).
[20] J. Shao, S. V. Buldyrev, L. A. Braunstein, S. Havlin, and H. E. Stanley, Phys. Rev. E **80**, 036105 (2009).
[21] M. E. J. Newman, Phys. Rev. E **66**, 016128 (2002).
[22] B. Bollobás, *Random Graphs* (Academic, London, 1985).

[23] *Fractals and Disordered Systems*, edited by A. Bunde and S. Havlin (Springer, Berlin, 1996).

[24] E. A. Leicht and R. M. D'Souza, "Percolation on interacting networks," e-print arXiv:cond-mat/0907.0894.

[25] L. K. Gallos, R. Cohen, P. Argyrakis, A. Bunde, and S. Havlin, Phys. Rev. Lett. **94**, 188701 (2005).

[26] A. A. Moreira, J. S. Andrade Jr., H. J. Herrmann, and J. O. Indekeu, Phys. Rev. Lett. **102**, 018701 (2009).

[27] H. Hooyberghs, B. VanSchaeybroeck, A. A. Moreira, J. S. Andrade Jr., H. J. Herrmann, and J. O. Indekeu, Phys. Rev. E **81**, 011102 (2010).

[28] X. Huang, J. Gao, S. V. Buldyrev, S. Havlin, and H. E. Stanley, "Robustness of interdependent networks under targeted attack," e-print arXiv:1010.2160.

[29] J. Gao, S. V. Buldyrev, S. Havlin, and H. E. Stanley, "Robustness of a network of networks," e-print arXiv:1010.5829.