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ENTRAINMENT COMPETITION IN COMPLEX NETWORKS

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The response of a random and modular network to the simultaneous presence of two frequencies is considered. The competition for controlling the dynamics of the network results in different behaviors, such as frequency changes or permanent synchronization frustration, which can be directly related to the network structure. From these observations, we propose a new method for detecting overlapping communities in structured networks.

Keywords: Small-world network; synchronization; competition.

1. Introduction

In natural complex systems the importance of how the collectivity of dynamical components responds and processes the external information is well known. Often, several different stimuli influence simultaneously the system and compete for the control of the dynamics, such as simultaneous sensorial signals acting on the neural system, or social groups exposed to several political, fashion or advertisement tendencies. In these processes, the dynamical units and their relationship with the underlying topolgy will be crucial in the final result. This kind of competitive dynamics has been studied in regular ensembles of oscillators [Leyva *et al.*, 2003], and recently the authors have considered the case for complex networks [Li *et al.*, 2008].

In order to explore more deeply this phenomenon, in this work, we study the competitive dynamics in a complex unstructured network (Sec. 2) and a modular one (Sec. 3) with two domains of synchronization. The behavior of each node is contrasted with its topological state. From the results, it can be inferred how the topology helps or prevents the ensemble to process incoming information. Finally, we propose a functional definition of the overlapping community in a modular network, as well as a specific detection algorithm [Li *et al.*, 2008].

2. Unstructured Network Structure

For our study, we first consider an initial graph $\mathcal{G}_0$ of n_0 bidirectionally coupled Kuramoto phase oscillators [Kuramoto, 1984], in which each node is randomly connected with probability $p = \ln(n_0)/n_0$ to the rest. Once this structure is defined, the coupling strength $d_{\rm net}$ is fixed so that this initial graph does not display a phase synchronized motion. Over this random core, a network formed only by links to two pacemaker oscillators is progressively grown. The pacemakers have frequencies, ω_1 and ω_2 respectively, which are going to compete for controlling the dynamics of $\mathcal{G}_0$.

The pacemakers ω_1 (ω_2) can establish links only with the first $i = 1, \ldots, n_0/2$ (last $i = n_0/2 + 1, \ldots, n_00$) nodes in $\mathcal{G}_0$. The new links have to be unidirectional to preserve the pacemaker character, and therefore, they result in a driving force for $\mathcal{G}_0$. The external links have coupling strength d_p. Once the growing process is complete, the final network has the structure illustrated in Fig. 1(a). The dynamics is described by [Sendina-Nadal *et al.*, 2008; Kuramoto, 1984]:

$$\dot{\phi}_i = \omega_i + \frac{d_{\rm net}}{(k_i + k_{p_i})} \sum_{j=1}^{n_0} a_{ij} \sin(\phi_j - \phi_i) + \frac{d_p k_{p_i}}{(k_i + k_{p_i})} \sin(\phi_p - \phi_i) \tag{1}$$

where i runs from 1 to n_0, k_i is the inner degree of the ith oscillator, i.e. the number of connections with other nodes in $\mathcal{G}_0$. k_{p_i} is its external degree, defined as the number of unidirectional connections with the forcing oscillator p, where $p = 1$ if $i < (n_0/2)$ and $p = 2$ otherwise. The p pacemaker phase is $\phi_p = \omega_p t$, with $p = 1, 2$.

The natural frequencies of the phase oscillators in $\mathcal{G}_0$ are $\{\omega_{0i}\}$, uniformly distributed within the range 0.5 ± 0.25. The coefficients $\{a_{ij}\}$ are the $n_0 \times n_0$ elements of the adjacency matrix $\mathbf{A} = (a_{ij})$, describing the structure of the connections in $\mathcal{G}_0$, with $a_{ij} = 1$ if the oscillators i and j are connected and $a_{ij} = 0$ otherwise.

A very important point is the selection criteria through which the pacemakers establish new links with $\mathcal{G}_0$. We here consider a dynamical criterion fully driven to enhance phase entrainment: when a new link is created, it is attached preferentially to that node in $\mathcal{G}_0$ whose instantaneous phase holds more closely to the phase condition: $\min_{i=1,\ldots,n_0} |\delta - \Delta\theta_i \bmod 2\pi|$ where $\Delta\theta_i = \phi_i(t) - \phi_p(t)$ and δ is a constant, that we fix in the following as $\delta = \pi$ [Sendina-Nadal *et al.*, 2008]. We choose d_p enough to entrain both halves of $\mathcal{G}_0$ to the corresponding pacemaker frequency. In the following, we will call C_1 (C_2) to the $\mathcal{G}_0$ nodes entrained by ω_1 (ω_2).

2.1. *Dependence on* $d_{\rm net}$

The previous procedure prepares our network $\mathcal{G}_0$ to have two domains of synchronization due to the influence of the external pacemakers. Since the inner coupling strength $d_{\rm net}$ is small, both frequencies do not yet affect each other. Then, at this point, we increase the value of $d_{\rm net}$ in order to generate a competitive dynamics between the external frequencies for synchronizing the most part of the network.

The results can be observed in Fig. 2, where we plot the temporal moving average of the individual frequencies of all $\mathcal{G}_0$ nodes, for a network with $n_0 = 200$ (and therefore $C_1 = C_2 = 100$), $\omega_1 = 0.3$ and $\omega_2 = 0.7$. For increasing values of $d_{\rm net}$, a small

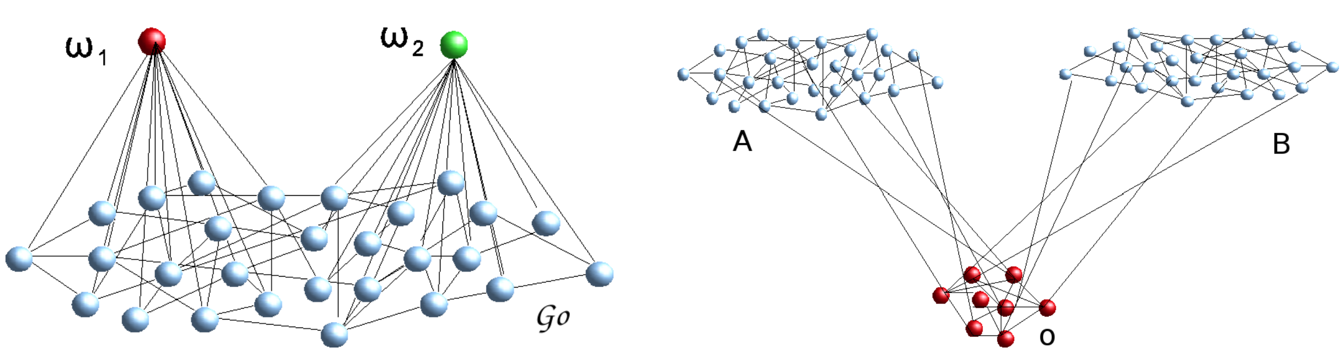


Fig. 1. Network designs to study competition between frequency domains: (a) random unstructured network $\mathcal{G}_0$ controlled by pacemakers ω_1, ω_2, (b) structured network with communities A, B overlapping by the module O.

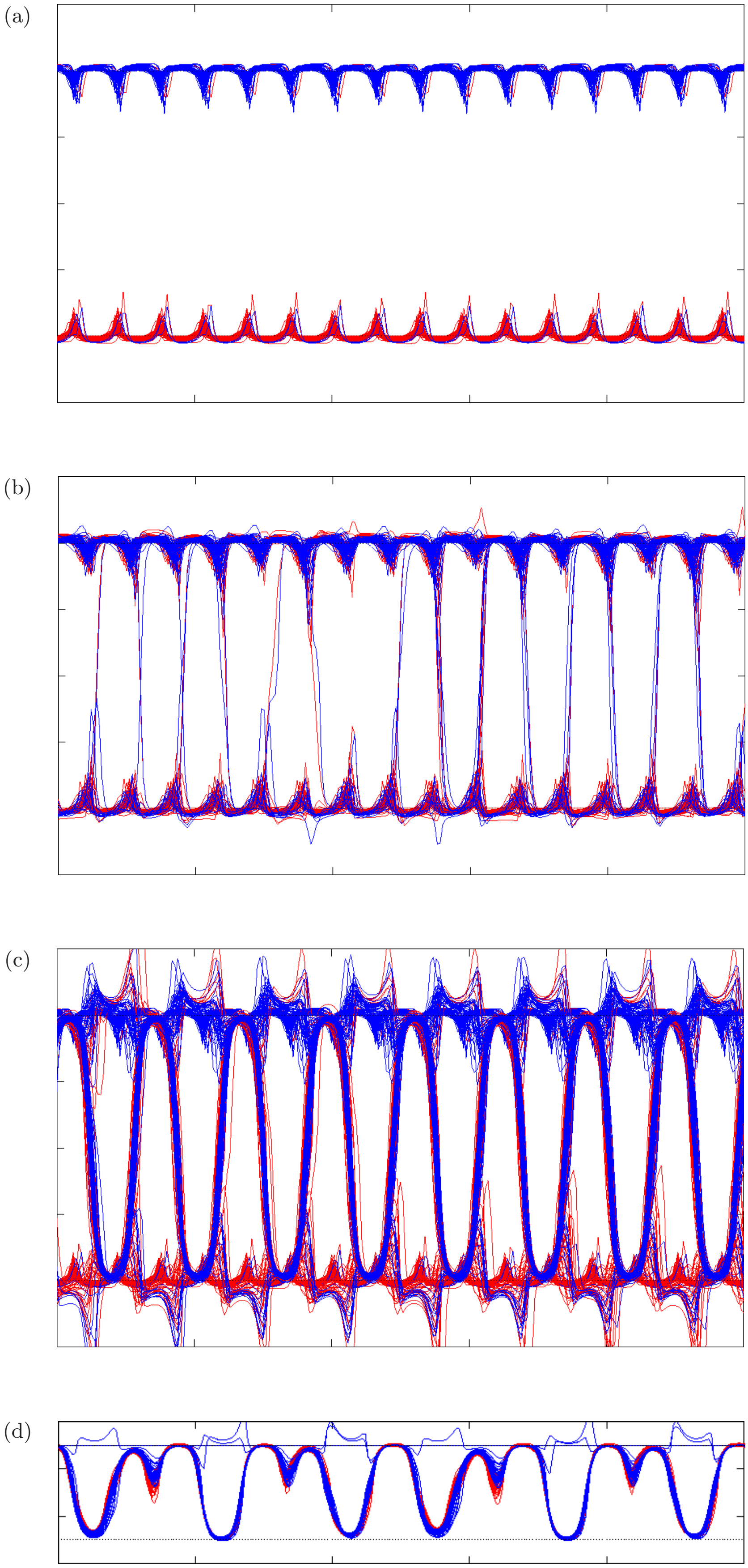
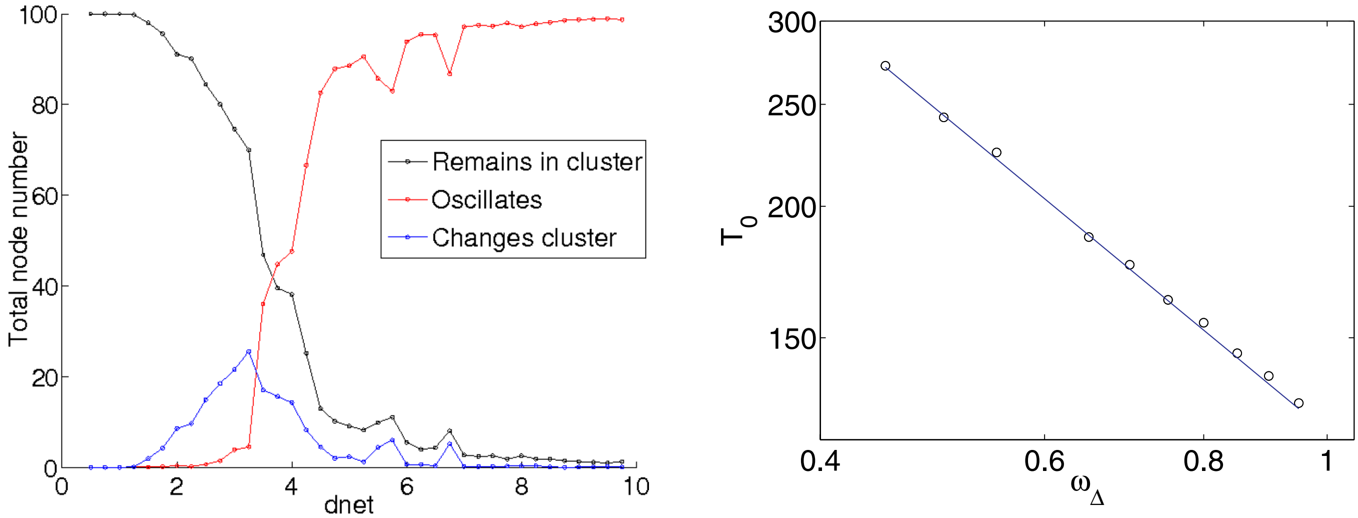


Fig. 2. Temporal moving average of the $\mathcal{G}_0$ nodes frequencies, for different $d_{\rm net}$ values: (a) 1.5, (b) 3.25, (c) 5.75, (d) 9.75.



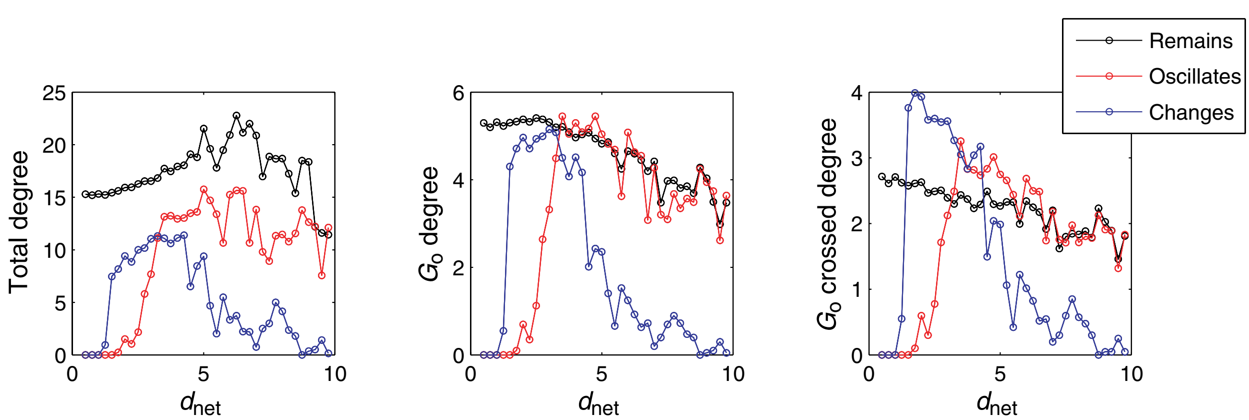
(a) (b)

Fig. 3. (a) Average number of nodes of a cluster that remain in the same cluster (black), changes to the other cluster (blue), or permanently oscillates between ω_1 and ω_2 (red). Total cluster size is 100 nodes. (b) Log–log plot of the oscillation period of the overlapping cluster T_o as a function of $\omega_\Delta = \omega_2 - \omega_1$.

fraction of nodes initially in C_1 changes cluster and synchronizes to C_2, and vice versa [Fig. 2(a)]. But more interestingly, for further increase of $d_{\rm net}$, a group of nodes start oscilling alternatively between both frequencies [Fig. 2(b)]. The oscillating nodes come from C_1 and C_2 without a clear preference. For even higher values of $d_{\rm net}$, the oscillation cluster becomes bigger [Fig. 2(c)], and finally, for sufficiently large $d_{\rm net}$, the whole $\mathcal{G}_0$ oscillates between both frequencies [Fig. 2(d)].

We can see this process more quantitatively in Fig. 3, where as $d_{\rm net}$ changes, we plot the average number of the nodes from a cluster that remains in the same cluster (in black), changes to the other cluster (in blue), or permanently oscillates between ω_1 and ω_2 (in red). Here it can be seen that only for intermediate values of $d_{\rm net}$ we find nodes that permanently change cluster. This points out to a strong influence of the topology on the individual node behavior.

The oscillation frequency of the overlapping cluster depends linearly on the entrainment frequencies difference $\Delta\omega = \omega_1 - \omega_0$, as can be observed in Fig. 3(b), where we show equivalently a log-log plot of the oscillation period of the overlapping cluster T_o as a function of ω_Δ. We see that the data fits well to a dependence $T_o \propto 1/\omega_\Delta$, which is usually a signal of competitive dynamics.



(a) (b) (c)

Fig. 4. Average degree of the nodes that remain, change or oscillate, as a function of: (a) total final degree, including pacemaker links, (b) inner degree (links inside $\mathcal{G}_0$) and (c) inner crossed degree (with nodes belonging to the other cluster). All the results are averaged over 10 realizations.

2.2. *Influence of the topology*

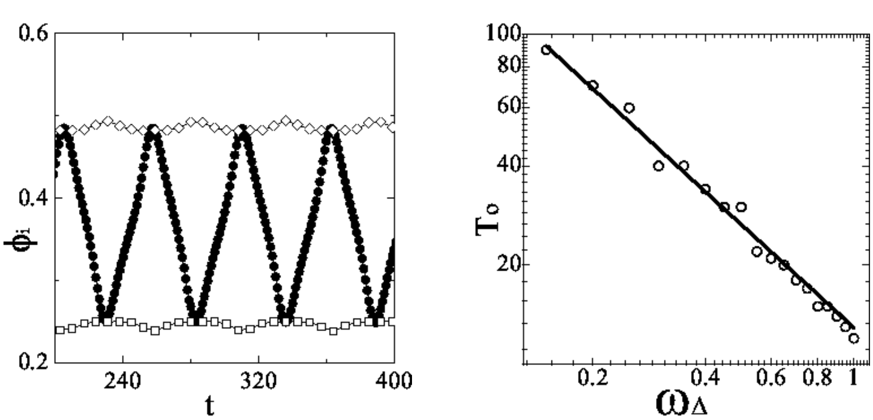
Once we have observed how the competitive dynamics works, we want to know how the topology of $\mathcal{G}_0$ determines the response. In particular, we study the correlations between the degree of each node and its dynamical behavior when the competition is working. In Fig. 4, we plot the average degree of the nodes that remain, change or oscillate, as a function of different topological measures: (a) the total degree $k_i + k_{p_i}$, including pacemaker links, (b) the inner degree k_i, taking into account only links inside $\mathcal{G}_0$, and finally in (c) only the initial $\mathcal{G}_0$ crossed degree, i.e. the links with nodes belonging to the other cluster of $\mathcal{G}_0$. It can be seen that, as expected, a high total degree is needed for remaining in the initial own cluster, since it assures a large number of external links, and therefore, these nodes are very influenced by the corresponding pacemaker. On the other hand, the nodes that change cluster have few external links, but a relatively strong connection to the other cluster. The oscillating nodes are well connected to both clusters, but their connection to the pacemaker is not very intense. These measurements show a clear correlation between the dynamical behavior observed and the topological state of each node.

3. Structured Network

Up to now we have studied the behavior of interfaces as the result of the competition of dynamical domains implemented in unstructured networks. However, it is even more interesting the case in which the dynamical domains correspond to a module in a structured network. So far, definitions of network communities found in literature led essentially to a topological partition into components such that each node belongs to and only to one of the components of the partition [Girvan *et al.*, 2002; Guimerá *et al.*, 2005]. However, our results let us suppose that in those cases in which two modules overlaps [Palla *et al.*, 2005], this group should reveal its condition in a dynamical or functional way.

In order to study this case of competition for modular graphs in the absence of external forcing [$d_p = 0$ and $k_{p_i} = 0\ \forall\, i$ in Eq. (1)], we construct a network with two communities A and B, each one with 50 densely and randomly connected nodes ($\langle k \rangle = 16$ inside community). Both of them overlap through a small module O made of five nodes with three random links to A and B, as illustrated in Fig. 1(b). All nodes in these three communities are associated with a phase oscillator following Eq. (1). This equation is integrated for an initial distribution of frequencies such that nodes in A and B have frequencies uniformly distributed in the intervals 0.25 ± 0.25 and 0.5 ± 0.25 respectively, while nodes in O have frequencies uniformly distributed in the range 0.375 ± 0.05, that is, around the mean frequency of the frequency distributions of A and B.

In Fig. 5(a) we show that all nodes in the main modules A and B are synchronized to the mean of their respective original frequency distribution, while O behaves as dynamical interface and therefore displays an instantaneous frequency that



(a) (b)

Fig. 5. (a) Instantaneous frequencies $\dot{\phi}_i(t)$ versus time from simulation of Eq. (1) with $d = 0.1$ (other parameters and stipulations are reported in the text). Squares, diamonds and full circles represent respectively nodes belonging to A, B and O. (b) Log–log plot of the switching period T_O of the oscillations in the frequency of the nodes in O versus the frequency difference ω_Δ. The solid line represents a linear fit with slope -1.002 ± 0.007.

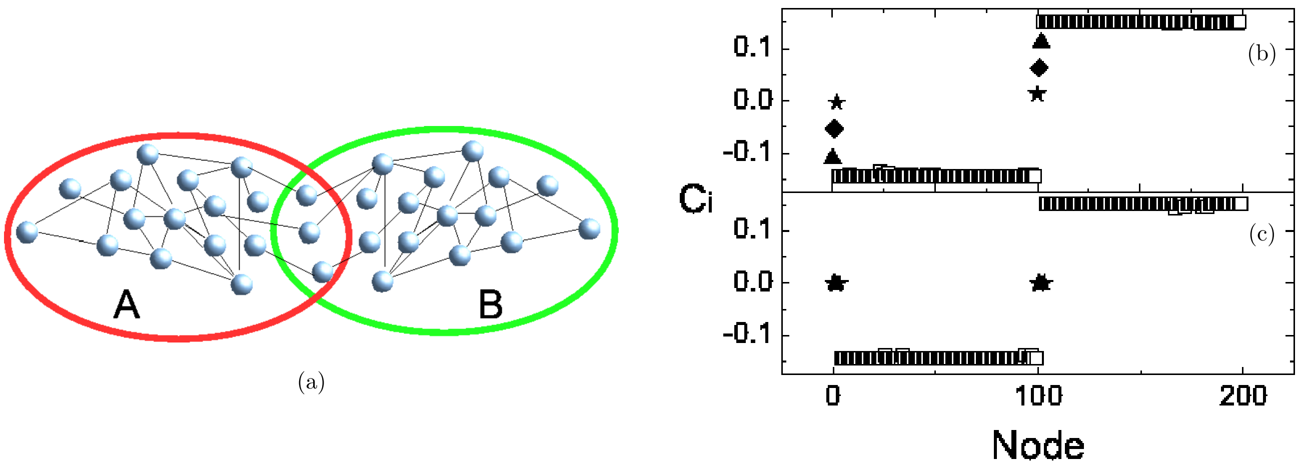


Fig. 6. (a) Illustration of the constructed modular network, where the intersection between the two circles (communities) represents the overlapping community. C_i (see text for definition) versus node index, where overlapping nodes $1, 2, 3$ $(101, 102, 103)$, labeled respectively with triangle, diamond and star, have: (b) asymmetrical, (c) symmetrical connections with nodes in main clusters A and B.

synchronizes alternatively with modules A and B. In Fig. 5(b) we plot the period of the frequency oscillations of O, which as can be seen scales linearly with the frequency difference between the two communities A and B, as in the unstructured case in Sec. 2.

3.1. *Detecting overlapping communities*

From the previous results we can derive an effective way to detect overlapping communities, defined dynamically as the set of nodes that, instead of following the constant frequency of their nominal communities, switch between two or more modules, and therefore they cannot be considered as a functional part of any of them.

In order to exemplify this idea, let us generate a network with two very dense large moduli (A and B of 100 nodes each), where a small group of nodes form links with sites in both communities (see illustration in Fig. 6(a)). To identify the overlapping nodes, we introduce the quantity:

$$C_i = \mathrm{sgn}[\dot{\phi}_i(t) - \bar{\omega}] \min_t \{|\dot{\phi}_i(t) - \bar{\omega}|\}$$

where $\bar{\omega}$ is the mean of the two averaged frequencies assigned to the two communities. Parameter C_i measures how close in time the dynamics of node i is to $\bar{\omega}$, and therefore when the node belongs to a synchronization interface $C_i \to 0$. The results are shown in Figs. 6(b) and 6(c) for two different arrangements of the overlapping community: asymetrically [Fig. 6(b)] and symmetrically connected overlapping nodes [Fig. 6(c)]. In both cases, two large synchronized clusters are identified very far from the overlapping synchronization, corresponding to those nodes performing unambiguous tasks and correctly classified by the usual modular partition algorithms. Simultaneously, parameter C_i evidences a group of nodes located significantly out of the two main clusters (thus identifying the overlapping community). In Fig. 6(b), each overlapping node gives rise to a different value of C_i in correspondence to its specific degree of overlapping due to the asymmetrical connectivity with both main clusters. In Fig. 6(c), all nodes inside the overlapping cluster are identified as a whole and feature the same value of $C_i = 0$ due to their symmetrical connections. The method can be succesfully applied in real networks [Li *et al.*, 2008].

4. Conclusions

We have shown that a complex network of phase oscillators may display interfaces between domains (clusters) of synchronized oscillations. The behavior of these interfaces are considered for unstructured and structured (modular) networks. The results lead us to propose a functional definition of overlapping communities in modular networks, and to develop a method to systematically obtain information on overlapping nodes in both artificial and real world modular networks.

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