

# Optimization of network robustness to random breakdowns

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## Abstract

We study network configurations that provide optimal robustness to random breakdowns for networks with a given number of nodes  $N$  and a given cost—which we take as the average number of connections per node  $\langle k \rangle$ . We find that the network design that maximizes  $f_c$ , the fraction of nodes that are randomly removed before global connectivity is lost, consists of  $q = [(\langle k \rangle - 1)/\sqrt{\langle k \rangle}] \sqrt{N}$  high degree nodes (“hubs”) of degree  $\sqrt{\langle k \rangle N}$  and  $N - q$  nodes of degree 1. Also, we show that  $1 - f_c$  approaches 0 as  $1/\sqrt{N}$ —faster than any other network configuration including scale-free networks. We offer a simple heuristic argument to explain our results.

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## 1. Introduction

Recently there has been much interest in determining network configurations which are robust against various types of attacks [1–9]. While there have been studies of complex combinations of different types of attacks, surprisingly, there has been no focused analysis of the elementary case of robustness against simple random breakdowns or attacks. As a measure of the robustness of the network, we study the quantity  $f_c$ , the fraction of nodes that are randomly removed before global connectivity is lost.

We first study simple random networks. Simple networks contain no self loops or multiple edges neither of which add to the robustness of a network to random removal of nodes. For simple random networks we can determine the optimal network configuration analytically. Randomly constructed networks, however, may have disconnected components, so we also consider networks constructed in a way that ensures they consist initially of a single cluster of connected nodes. These networks are degree correlated. For degree correlated networks, there currently exist no closed-form expressions with which we can determine  $f_c$  analytically, so we study them using Monte Carlo simulations. We find that the optimal configuration for both the randomly constructed and the degree correlated networks consists of  $q \sim \sqrt{N}$  high degree nodes (*hub* nodes) of degree  $k_2 \sim \sqrt{N}$  and  $N - q$  nodes of degree 1 (*leaf* nodes).

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## 2. Uncorrelated networks

### 2.1. Theory

We first treat simple random networks. It is known [10–13] that for any desired random degree distribution, simple networks can be created only if  $P(k) = 0$  for  $k$  greater than the *structural cutoff*

$$K_s \equiv \sqrt{\langle k \rangle N}. \tag{1}$$

So we must limit our networks to those with maximum degree less than  $\sqrt{\langle k \rangle N}$ . For networks with this constraint we can use the equation [3]

$$f_c = 1 - \frac{1}{\kappa - 1}, \tag{2}$$

where

$$\kappa \equiv \frac{\langle k^2 \rangle}{\langle k \rangle} \tag{3}$$

to determine  $f_c$ . The domain of validity of Eq. (3) is discussed in detail in Ref. [14].

Since we fix  $\langle k \rangle$ , maximizing  $f_c$  is equivalent to maximizing  $\langle k^2 \rangle$ . We must maximize

$$h(P) \equiv \sum_{k=1}^{K_s} k^2 P(k) \tag{4}$$

under the following constraints:

$$P(k) \geq 0, \tag{5}$$

$$\sum_{k=1}^{K_s} k P(k) = \langle k \rangle, \tag{6}$$

$$\sum_{k=1}^{K_s} P(k) = 1. \tag{7}$$

We first show that there can be no more than two unique values of  $k$  at which  $P(k)$  is non-zero if  $h(P)$  is to be maximized. Assume that there are  $m > 2$  non-zero values  $P(k_1), P(k_2), P(k_3) \dots P(k_m)$  needed to maximize  $h(P)$ . Using the method of Lagrange multipliers [15] we can write

$$\frac{\partial \left( \sum_{i=1}^{K_s} k_i^2 P(k_i) \right)}{\partial P(k_j)} + \lambda_1 \frac{\partial \left( \sum_{i=1}^{K_s} k_i P(k_i) - \langle k \rangle \right)}{\partial P(k_j)} + \lambda_2 \frac{\partial \left( \sum_{i=1}^{K_s} P(k_i) \right)}{\partial P(k_j)} = 0 \tag{8}$$

or

$$k_j^2 + \lambda_1 k_j + \lambda_2 = 0 \quad (1 \leq j \leq m), \tag{9}$$

where  $\lambda_1$  and  $\lambda_2$  are constants. Solving (9) we find at most only two unique solutions for the values of  $k_j$ .

Analyzing the problem now with only two values  $k_1$  and  $k_2$  for which  $P(k)$  are non-zero, we find that  $h(P)$  is maximized when  $k_1$  and  $k_2$  take on the boundary values

$$k_1 = 1, \tag{10a}$$

$$k_2 = K_s, \tag{10b}$$

and

$$P(k_1) = 1 - \frac{\langle k \rangle - 1}{\sqrt{\langle k \rangle}} \frac{1}{\sqrt{N}}, \tag{11a}$$

$$P(k_2) = \frac{\langle k \rangle - 1}{\sqrt{\langle k \rangle}} \frac{1}{\sqrt{N}}. \quad (11b)$$

For these values  $1 - f_c$  assumes its minimal value

$$(1 - f_c)_{\min} = \frac{\langle k \rangle \sqrt{\langle k \rangle N}}{(\langle k \rangle - 1)(1 - \langle k \rangle N + \sqrt{\langle k \rangle N})}. \quad (12)$$

For large  $N$ ,

$$(1 - f_c)_{\min} \sim \frac{\sqrt{\langle k \rangle}}{(\langle k \rangle - 1)} \frac{1}{\sqrt{N}}. \quad (13)$$

## 2.2. Simulations

We next perform Monte Carlo simulations to test the results found above. We consider the degree distribution that represents a network of  $q$  hub nodes and  $N - q$  leaf nodes,

$$P(k) = \begin{cases} \frac{N - q}{N}, & k = 1, \\ \frac{q}{N}, & k = k_2, \\ 0 & \text{otherwise,} \end{cases} \quad (14)$$

where

$$k_2 = \frac{(\langle k \rangle - 1)N + q}{q}. \quad (15)$$

Our aim is to find the value of  $q$  which maximizes the robustness of the network.

We create networks using the method described in Ref. [16]. We then randomly delete nodes in the network and after each node is removed, we calculate  $\kappa$ . We use the criterion

$$\kappa < 2 \quad (16)$$

for loss of global connectivity [3,4,6,16]. When  $\kappa$  becomes less than 2 we record the number of nodes  $n_r$  removed up to that point. This process is performed for many realizations of random graphs with the degree distribution of Eq. (14) and, for each graph, for many different realizations of the sequence of random node removals. The threshold  $f_c$  is defined as

$$f_c \equiv \frac{\langle n_r \rangle}{N}, \quad (17)$$

where  $\langle n_r \rangle$  is the average value of  $n_r$ .

In Fig. 1(a), we plot  $1 - f_c$  versus  $q$  for  $N = 10^2, 10^3, 10^4$  and  $10^5$  and  $\langle k \rangle = 2$ . In Fig. 2(b) we plot the location of the minima  $q_{\min}$  versus  $N$ . As expected  $q_{\min}$  scales with  $N$  as  $N^{0.5}$  and as shown in Fig. 2(c) the minimum values of  $1 - f_c$  scale as  $N^{-0.5}$ .

Also shown in Fig. 1(a) are plots for approximations to  $f_c$ ,  $f_c^{\text{high}}$  and  $f_c^{\text{low}}$ , which we expect to be valid, respectively, for high and low values of  $q$ . We will use these approximations as another way to show that  $q_{\min}$  and  $(1 - f_c)_{\min}$  scale as found above. The approximations are determined as follows:

- (i) When  $q \sim N$  (i.e., the network is homogeneous), we expect Eq. (2) to hold, so  $f_c^{\text{high}} = 1 - 1/(\kappa - 1)$ . For general  $\langle k \rangle$ , using the distribution in Eqs. (14), we find for  $N \gg q \gg 1$

$$f_c^{\text{high}} = 1 - \frac{q}{(\langle k \rangle - 1)N}. \quad (18)$$

- (ii) As found in Ref. [14], Eq. (2) is not valid for small  $q$ . We must use an approximation based on the fact that for small  $q$  the network loses global connectivity when all  $q$  high degree nodes are removed. To first order

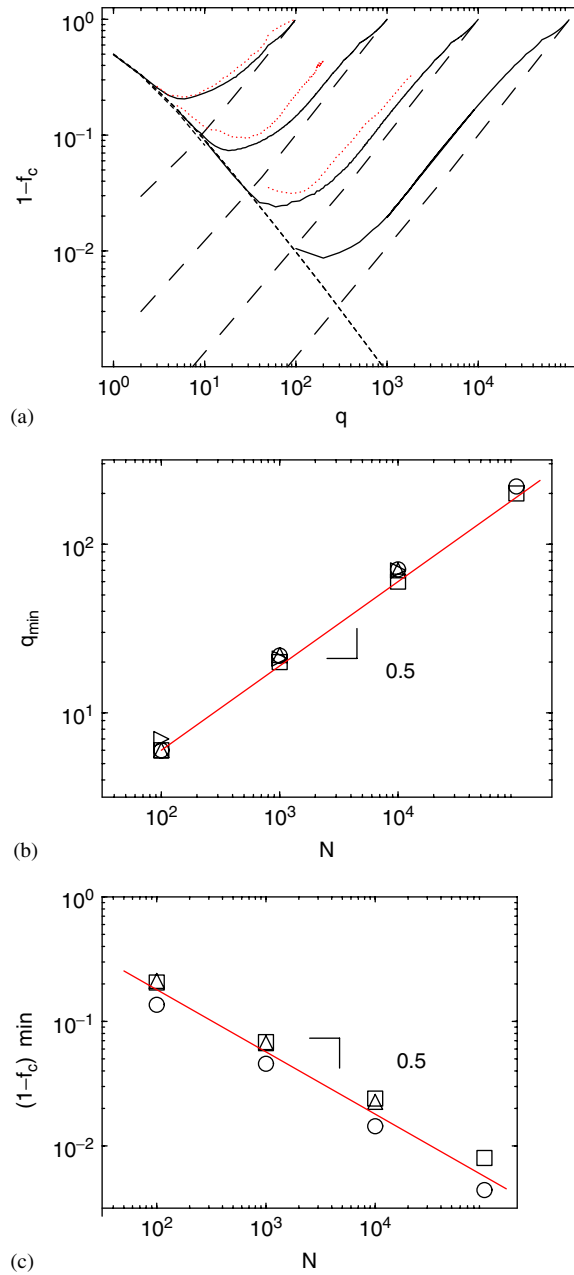


Fig. 1. For  $\langle k \rangle = 2$  and for (from left to right)  $N = 10^2, 10^3, 10^4$  and for random networks only  $10^5$  (a)  $1 - f_c$  versus number of hubs  $q$ . The solid and dotted lines represent uncorrelated and correlated networks, respectively. Dashed lines(short) are approximation  $f_c^{low}$ ; dashed lines(long) are approximation  $f_c^{high}$ . (b) Values of  $q, q_{min}$ , at which  $1 - f_c$  is minimal versus  $N$ . Squares and triangles represent uncorrelated and correlated networks, respectively; circles represent  $q^*$  the value of  $q$  at which the approximations  $f_c^{high}$  and  $f_c^{low}$  intersect. (c) Minimum values of  $1 - f_c$  versus  $N$ . Squares and triangles represent uncorrelated and correlated networks, respectively; circles represent the values of  $(1 - f_c)$  at  $q = q^*$ .

in  $1/q$  [14]

$$1 - f_c^{low} = 1 - \frac{1}{q}. \tag{19}$$

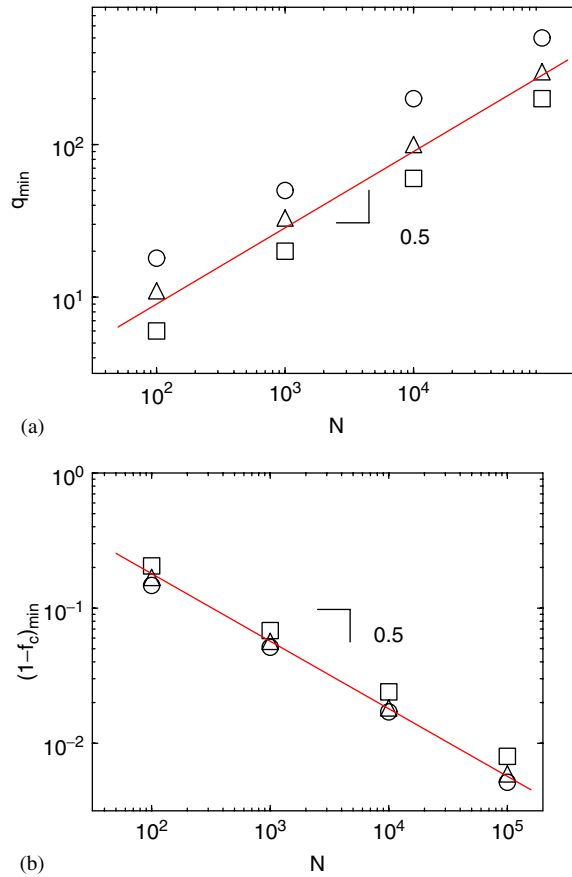


Fig. 2. (a) Values of  $q$ ,  $q_{\min}$ , at which  $1 - f_c$  is minimal versus  $N$ . Squares, triangles and circles represent network with  $\langle k \rangle = 2, 3$ , and  $4$ , respectively. (b) Minimum values of  $1 - f_c$  versus  $N$ . Squares, triangles and circles represent networks with  $\langle k \rangle = 2, 3$ , and  $4$ , respectively.

Equating Eqs. (18) and (19) we find the value of  $q$  at which the approximations intersect

$$q^* = \sqrt{\langle k \rangle - 1} \sqrt{N}. \tag{20}$$

From the fact that  $q^*$  scales like  $\sqrt{N}$ , we conclude that all characteristic values including the location of the minimum of  $(1 - f_c)$  scale like  $\sqrt{N}$  with a prefactor dependent on  $\langle k \rangle$ .

From Eqs. (19) and (20) we find for large  $N$ ,

$$1 - f_c^* = \frac{1}{\sqrt{\langle k \rangle - 1}} \frac{1}{\sqrt{N}}, \tag{21}$$

where  $f_c^*$  is the value of value of  $f_c$  where the approximations intersect. The scaling of  $q^*$  and  $1 - f_c^*$  are shown in Figs. 1(b) and (c).

We next study the effect of changing  $\langle k \rangle$ . Figs. 2(a) and (b) contain plots of  $q_{\min}$  and  $(1 - f_c)_{\min}$ , respectively, for  $\langle k \rangle = 2, 3$ , and  $4$ . We note that the scaling is independent of  $\langle k \rangle$  with only a change in the prefactor.

### 3. Correlated networks

In Fig. 3(a) we show an example of a randomly created graph. Note that, because the graph is created randomly, there are some disjoint portions of the graph consisting of pairs of nodes connected to each other.

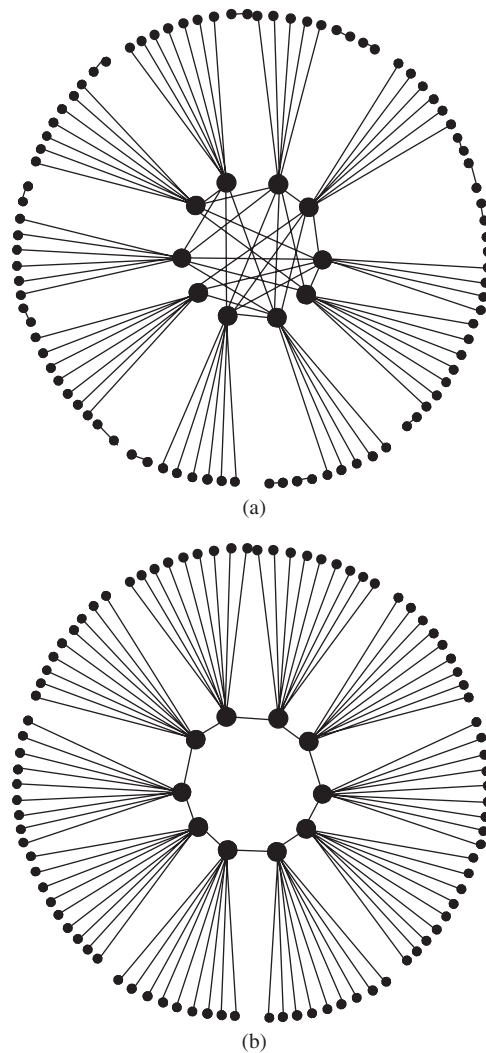


Fig. 3. Examples of 100 node networks with degree distribution given by Eqs. (14) with  $\langle k \rangle = 2$ : (a) uncorrelated network. Note that there are disconnected pairs of nodes of degree 1; (b) correlated network in which each degree 1 node is connected to a high degree node.

Thus the network does not consist of a single connected component. We now study correlated networks which do not have this shortcoming by disallowing connections between degree 1 nodes so that the resulting network is a single cluster (see Fig. 3(b) which has the same degree distribution as Fig. 3(a)).

For correlated networks, the criteria for network collapse is [17]

$$\det(\mathbf{A}) = 0, \quad (22)$$

where  $\mathbf{A}$  is a matrix containing elements  $A_{j,k} = ke_{j,k} + q_j\delta_{i,j}$  with  $e_{j,k}$  the joint probability of the remaining degrees [18] of the two vertices at either end of a randomly chosen edges and with  $q_k$  the probability of the remaining degree of a single vertex at the end of a randomly chosen edge.

We create networks having the degree distribution of Eq. (14) with  $\langle k \rangle = 2$  but with the constraint that leaf nodes cannot be connected to each other. We proceed as for uncorrelated networks except that after removal of an edge instead of calculating  $\kappa$  we calculate  $\det(\mathbf{A})$  and note the number of nodes removed before  $\det(\mathbf{A}) = 0$ .

In Fig. 1(a) we plot  $1 - f_c$  versus  $q$  for  $N = 10^2, 10^3$  and  $10^4$  [19]. We note that the plots are similar to but slightly higher than the corresponding plots for the random networks. In Fig. 1(b) we plot the values of  $q$  at which  $1 - f_c$  is minimal and see that they scale in a manner similar to the scaling of the positions of the minima for the random networks.

**4. Comparison with scale-free networks**

Scale-free networks with  $\lambda < 3$  are known to be very robust against random attack [1,3] with  $1 - f_c$  approaching zero as  $N \rightarrow \infty$ . Here, we determine the large  $N$  behavior of  $1 - f_c$  for scale-free networks for a given value of  $\langle k \rangle$  and compare the behavior with that of the optimal bimodal network.

We consider a scale-free degree distribution  $P(k) \sim k^{-\lambda}$  with  $m \leq k \leq K$ . For large  $K$  and  $2 < \lambda < 3$  [3],

$$\kappa = \frac{2 - \lambda}{3 - \lambda} m^{\lambda-2} K^{3-\lambda}. \tag{23}$$

Substituting in Eq. (2) and setting  $K = K_s$  we find that for large  $K$

$$1 - f_c \sim K^{3-\lambda} \sim N^{(\lambda-3)/2}. \tag{24}$$

Only in the limit of  $\lambda$  approaching 2 does  $1 - f_c \sim N^{-0.5}$  similar to Eq. (13). For  $\lambda < 2$ ,

$$\kappa = \frac{2 - \lambda}{3 - \lambda} K \sim \sqrt{N} \tag{25}$$

and

$$1 - f_c \sim \frac{1}{\sqrt{N}}, \tag{26}$$

but for  $\lambda \leq 2$ ,  $\langle k \rangle$  diverges with increasing  $K$ . Thus for a given value of  $\langle k \rangle$ ,  $1 - f_c$  for the optimal bimodal network always approaches 0 faster than the optimal scale-free network [20].

For completeness, to ensure that large variance is not a deficiency of the optimal network, we now study how the variance in  $f_c$  of the optimal network compares with the variance of the scale-free networks. Specifically in Fig. 4 we plot the standard deviation

$$\sigma = \frac{\sqrt{\langle (n_r - \langle n_r \rangle)^2 \rangle}}{N} \tag{27}$$

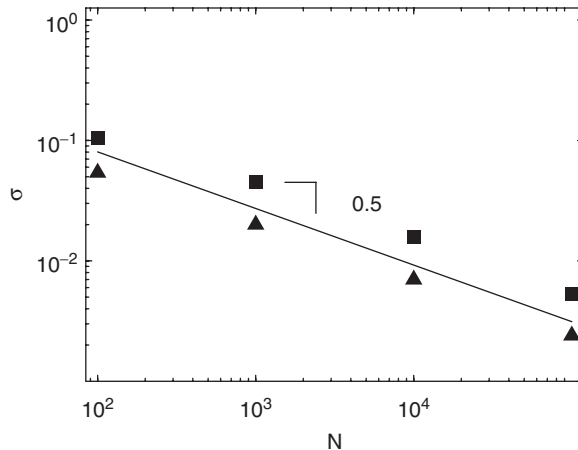


Fig. 4. Standard deviation in  $f_c$  versus  $N$ . Squares represent optimal bimodal configuration for  $\langle k \rangle = 2$ . Triangles represent scale-free configuration with  $\lambda = 2$ .

versus  $N$  for the optimal bimodal network with  $\langle k \rangle = 2$  and for a scale-free network with  $\lambda = 2$ . For the scale-free network with  $\lambda = 2$ ,  $1 - f_c \sim N^{-0.5}$  although it has a large value of  $\langle k \rangle$ . We see that the standard deviation of  $f_c$  of both networks decreases as  $N^{-0.5}$  with the scale-free network having a somewhat smaller prefactor than the optimal network. Thus the variance of  $f_c$  is not a deficiency of the optimal network.

## 5. Heuristic argument for optimal configuration

We now provide a heuristic argument for the optimal configuration which applies to random or correlated networks. As shown above, the configuration consists of  $q \sim \sqrt{N}$  high degree nodes (hubs) of degree  $\sqrt{\langle k \rangle N}$  and  $N - q$  nodes of degree 1. Intuitively, we suspect that the optimal configuration is one in which there are many leaf nodes with degree 1 connected to a network core composed of a much smaller number of highly connected hubs node. Removing a leaf node has only a minimal impact on the connectivity of the network while removing a hub has a much greater impact—but is much less probable. It is not obvious, however, how many hubs there should be. One might initially suppose that the most robust network would be a single hub node connected to all the remaining nodes (a *star* network). It is easy to show [14], however, that  $f_c = \frac{1}{2}$  for this network which is far from optimal. To determine the number of hubs we proceed as follows. Consider first that there are  $\langle k \rangle N$  connections available to construct the network. Let  $q$  denote the number of hubs. The number of connections needed to connect the hubs to the leaf nodes is  $2(N - q)$ . If we then make the argument that we want the hubs to form a complete graph using the remaining connections we have

$$q(q - 1) = \langle k \rangle N - 2(N - q). \quad (28)$$

Solving for  $q$  for large  $N$  we have

$$q \sim \sqrt{\langle k \rangle - 2} \cdot \sqrt{N} \quad (29)$$

and we again find that the number of hubs scales as  $\sqrt{N}$  in a manner similar to that implied by Eq. (11b),

$$q = \left( \frac{\langle k \rangle - 1}{\sqrt{\langle k \rangle}} \right) \sqrt{N}, \quad (30)$$

for the optimal network with a different prefactor.

## 6. Discussion and summary

We have shown analytically and confirmed numerically using Monte Carlo simulations that networks with bimodal degree distributions, with  $q \sim \sqrt{N}$  high-degree nodes (hubs) and  $N - q$  nodes of degree 1, are most robust to random breakdown. Also we have shown that  $1 - f_c$  approaches 0 as  $1/\sqrt{N}$ , faster than any other network configuration including scale-free networks. Finally, we have offered a simple heuristic argument which explains these results.

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- [19] Because of the computation complexity of finding the determinant of  $A$  we do not treat  $N = 10^5$  as we did for random networks.
- [20] We have also performed simulations that show that even when  $K > K_s$ ,  $1 - f_c$  does not approach 0 faster than  $1/\sqrt{N}$  for any value of  $\lambda$ .