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Percolation of spatially constraint networks

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Abstract – We study how spatial constraints are reflected in the percolation properties of networks embedded in one-dimensional chains and two-dimensional lattices. We assume long-range connections between sites on the lattice where two sites at distance r are chosen to be linked with probability $p(r) \sim r^{-\delta}$. Similar distributions have been found in spatially embedded real networks such as social and airline networks. We find that for networks embedded in two dimensions, with $2 < \delta < 4$, the percolation properties show new intermediate behavior different from mean field, with critical exponents that depend on δ. For $\delta < 2$, the percolation transition belongs to the universality class of percolation in Erdös-Rényi networks (mean field), while for $\delta > 4$ it belongs to the universality class of percolation in regular lattices. For networks embedded in one dimension, we find that, for $\delta < 1$, the percolation transition is mean field. For $1 < \delta < 2$, the critical exponents depend on δ, while for $\delta > 2$ there is no percolation transition as in regular linear chains.

Complex networks have attracted considerable attention in the last decade [1–12]. It has been realized that networks provide a very useful way to describe and better understand the collective behavior of complex systems composed of a large number of interacting entities. There are two network classes of particular interest: Erdös-Rényi (ER) random graphs [13] and Barabasi-Albert (BA) scale-free networks [14]. In ER networks, the distribution of the degrees k (number of links) of the nodes is a Poissonian ($P(k) \sim \lambda^k/k!$, where λ is the average degree), while in BA scale-free networks, the distribution follows a power law, $P(k) \sim k^{-\gamma}$, with γ typically between two and three. Both classes have interesting topological properties, considerably different from those of regular lattices. Both exhibit the "small world" effect meaning that their topological diameter increases slowly, either logarithmically or double logarithmically, with the system size [9,10].

When studying the properties of networks it is usually assumed that spatial constraints can be neglected. This is probably true for certain networks such as the WorldWideWeb (WWW) or the citation network where the real (Euclidean) distance does not play a role. In contrast, the spatial distance does play a role in the Internet [15,16], airline networks [17,18], human travel networks [12,19], wireless communication networks [20] and social networks [21,22], which are all embedded in two-dimensional space. It has recently been shown that these spatial constraints are important and in certain cases can significantly alter the topological properties of the networks [23–32].

In this letter, we study the robustness of spatially constrained networks embedded in one or two-dimensional space, by analyzing their percolation properties. Percolation is important since it can shed light also on epidemic spreading [33] and immunization strategies [34]. Here, we focus on spatially embedded ER networks where lattice nodes are connected to each other with a probability $p(r) \sim r^{-\delta}$, where r is the Euclidean distance between the nodes. The choice of a power law for the distance distribution is supported from findings in several real

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complex networks, such as the Internet [15,16], airline networks [17,18], human travel networks [12,19] and other social networks [21,22,25].

Here, we are interested in studying how in model networks δ influences the percolation properties of the embedded networks, such as the percolation threshold q_c, the size of the largest cluster M, its dependence on the system size (number of nodes N) and the cluster size distribution n_s at q_c. When the networks are embedded in two-dimensional space we find that, for $2 < \delta < 4$, the percolation properties show new intermediate behavior different from mean field with critical exponents that depend on δ. For $\delta < 2$, the percolation belongs to the universality class of percolation in ER networks (mean field), while for $\delta > 4$ it belongs to the universality class of percolation in two dimensional lattices. For networks embedded in linear chains we find that, for $\delta < 1$, the percolation properties are the same as mean field, while for $1 < \delta < 2$, we find new percolation exponents that depend on δ. For $\delta > 2$, there is no percolation transition, as in normal one-dimensional systems. We like to note that analogous δ regimes were found in these systems also for the dependence of the shortest path on N [28].

We focus on embedded networks where each node has roughly the same number of k links. To construct these networks we follow the algorithm presented in [28]. The nodes of the embedded network are the sites of a given regular lattice. First, we choose a node i randomly and assume that it is connected by k links to other k nodes. For one of those links, we select a length r with probability $\phi(r)$ ($\phi(r) = cp(r) = cr^{-\delta}$ for the linear chain and $\phi(r) = crp(r) = crr^{-\delta}$ for the square lattice where c is determined from the normalization condition). There are N_r nodes at the selected distance r from node i, and we pick randomly one of them and connect it to node i. As a result, $k-1$ links of node i are not assigned yet. Next, we repeat this process for another randomly chosen node, where again we establish one link. We continue this process where at each step one of the remaining links is connected, until all links are assigned or no more links can be assigned. Finally, we remove the rare multiple connections. Due to the generation process, the nodes of the final network do not all have the same degree, but the degree follows a narrow distribution with a mean $\bar{k}$ that depends on both k and δ. In the two-dimensional case, $k = 4$ produces networks with $3.6 < \bar{k} \leqslant 4$ for all δ, while in one dimension, $k = 4$ produces networks with $3.1 < \bar{k} < 4$ for all $\delta < 2$.

Next we study the percolation properties of these embedded network models. To this end, we randomly remove a fraction q of nodes and measure the size of the largest and second largest cluster as a function of q. The percolation transition q_c is estimated as that value of q where the second largest cluster on the network reaches a maximum. In fig. 1 we present four examples of the largest cluster ("giant component") at the percolation transition for networks embedded in a square lattice, for $\delta = 1.5$, $\delta = 2.5$, $\delta = 3.5$ and $\delta = 4.5$. By inspection one can see

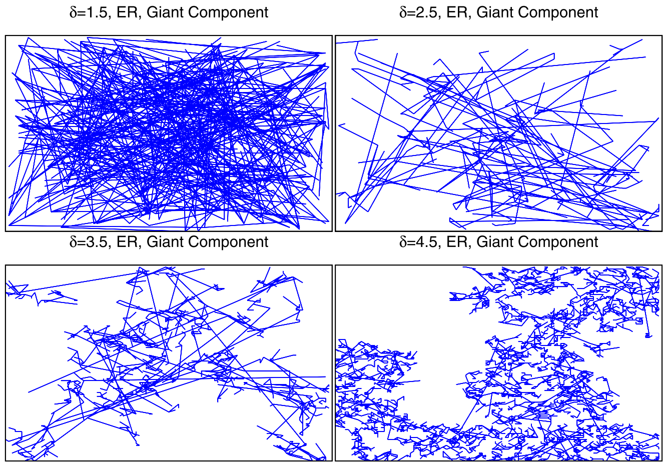

Fig. 1: (Color online) Giant component of ER networks embedded in a square lattice at the percolation threshold with $\delta = 1.5$, 2.5, 3.5 and 4.5. As δ increases, the giant component has less long-range connections and becomes more affected by the constraints of the embedding space.

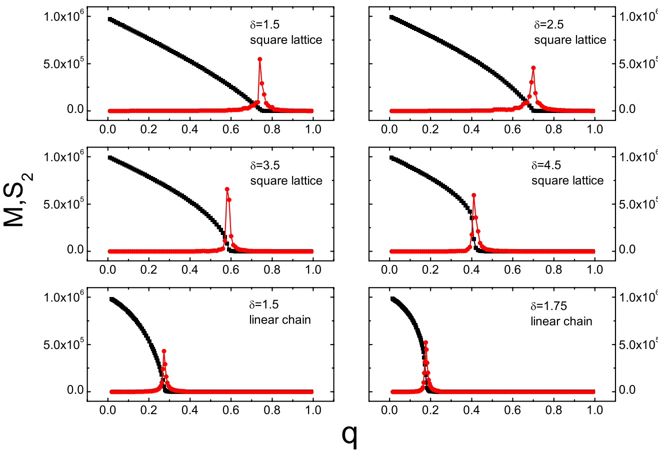

Fig. 2: (Color online) Size of the largest cluster M (black) and the second largest cluster S_2 (red) as a function of the removed fraction of nodes q for several values of δ. The vertical scales refer to M. For better visibility, we rescaled S_2 by appropriate factors. For networks embedded in a square lattice (upper 4 panels), the size of the second largest cluster reaches a maximum value at $q_c = 0.75, 0.70, 0.60$, and 0.41 for $\delta = 1.5, 2.5, 3.5$, and 4.5, respectively. For networks embedded in linear chains (2 lower panels), $q_c = 0.27$ and 0.17 for $\delta = 1.5$ and 1.75, respectively.

three qualitatively different structures appearing in the giant component. For $\delta = 1.5$ and 2.5, there is a large number of long links which are responsible for the connectivity of the largest cluster. For $\delta = 3.5$ the giant component is made up from several localized subgraphs which are connected to each other by few long links. Finally, for $\delta = 4.5$ the giant component is made by local short-range connections only, similar to site percolation in lattices.

To obtain q_c, we measure the size M of the largest cluster and the size S_2 of the second largest cluster (fig. 2)

Table 1: Critical exponents and thresholds for ER networks embedded in square lattices (upper table) and linear chains (lower table).

δ	τ	d_f/d (eq. (1))	d_f/d (eq. (2))	q_c
4.5	2.05	0.94	0.95	0.41
3.5	2.16	0.86	0.86	0.60
3.3	2.21	0.82	0.83	0.62
3.0	2.36	0.75	0.74	0.66
2.8	2.38	0.71	0.72	0.68
2.5	2.48	0.69	0.68	0.70
1.5	2.50	0.66	0.66	0.75

δ	τ	d_f/d (eq. (1))	d_f/d (eq. (2))	q_c
1.75	2.11	0.90	0.90	0.17
1.50	2.35	0.74	0.74	0.27

as a function of q, see fig. 2. When approaching q_c from below, the fraction of nodes in the largest cluster approaches zero, but the size of the second largest cluster reaches a maximum value [35]. Accordingly, by determining the size of the second largest cluster, we can determine the value of q_c. Table 1 shows the values of q_c for different values of δ. For $\delta = 4.5$, $q_c = 0.41$ is very close to the known value of site percolation in the square lattice, $q_c = 0.407$, while for $\delta = 1.5$, $q_c = 0.75$ is very close to the known result $q_c = 1 - 1/\bar{k}$ for percolation in ER networks. For smaller δ values, the network contains more long-range connections. As a consequence, it becomes more robust against node removal, and q_c increases.

Next we analyze the critical exponent that determines how at criticality the size M of the giant component scales with the number of nodes N of the original network. Figure 3 shows, in a double-logarithmic presentation, M *vs.* N in networks embedded in a square lattice and in a linear chain, for several δ values and several values of q in the vicinity of q_c. The figure shows that only at criticality there exists a power-law relation (straight line in the log-log plot) while above and below q_c the curves bend up and down, respectively. This feature is a characteristic of the percolation transition [35]. The fact that there is a strict power law only at q_c can also be used to determine q_c quite accurately. Since at q_c the largest cluster is expected to be a fractal, we obtain

$$M \sim N^{\frac{d_f}{d}}, \tag{1}$$

where d_f is the fractal dimension of the percolation cluster [35] and d is the dimension of the embedded network. Table 1 summarizes our results for d_f/d. We find that in the square lattice for $\delta < 2$, the networks are not influenced by the spatial constraints and show mean-field–like behavior, with $d_f/d \cong 0.66$ as expected for ER graphs ($d_f/d = 2/3$) [9,13]. For $\delta > 4$, where long-range connections are very rare, the networks are organized locally and $d_f/d = 91/96 \approx 0.95$ as for regular two-dimensional lattices [35]. In the intermediate regime

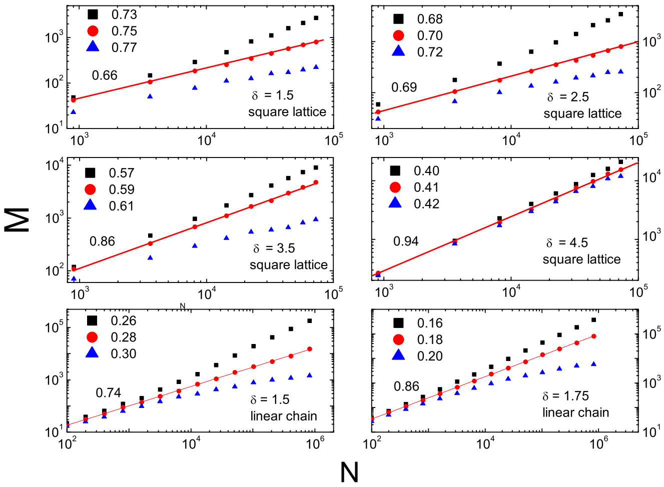

Fig. 3: (Color online) Size of the largest cluster M as a function of N for different values of δ and different fractions of removed nodes: squares ($q > q_c$), circles ($q = q_c$) and triangles ($q > q_c$).

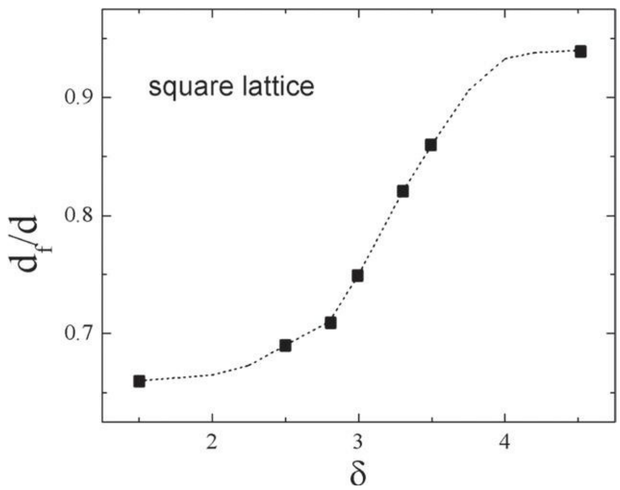

Fig. 4: The values of d_f/d at criticality for different values of δ in ER networks embedded in square lattices. The dashed line is only a guide for eyes. For $\delta > 4$, the percolation properties belong to the universality class of square lattice ($d_f/d = 91/96$), while, for $\delta > 2$, they belong to mean-field result ($d_f/d = 2/3$).

($2 < \delta < 4$), our results indicate the existence of new critical exponents (see also fig. 4). Figure 3 (lower panels) indicates, that a similar intermediate regime $1 < \delta < 2$ exists also in linear chains, where $d_f/d = 0.74$ and 0.9 for $\delta = 1.5$ and 1.75, respectively.

Next, to support these results, we apply an independent approach to evaluate d_f/d by analyzing the cluster size density distribution $n(s)$ at criticality, see fig. 5. The cluster size distribution is defined such that $n(s)\mathrm{d}s$ is the probability to find a cluster with size between s and $s + \mathrm{d}s$. At criticality, $n(s)$ is expected to scale as $n(s) \sim s^{-\tau}$ [35], where

$$\tau = 1 + \frac{d}{d_f}. \tag{2}$$

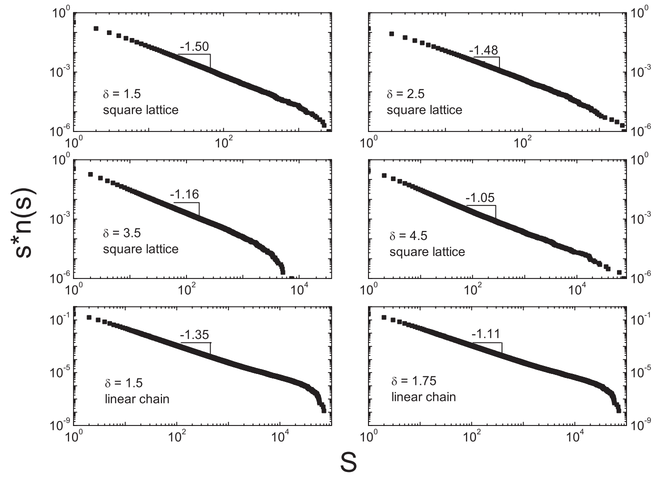

Fig. 5: Density distribution $n(s)$ of cluster sizes multiplied by s at criticality for different values of δ.

Accordingly, by measuring the exponent τ (slopes in fig. 5), we obtain an independent evaluation of d_f/d. The results are shown in table 1 and are in very good agreement with the values obtained from eq. (1). For networks embedded in a square lattice, with $\delta < 2$, we obtain the classical mean-field value $\tau = 2.5$ known for ER networks [35]. For $\delta > 4$, we obtain the same exponent as for percolation in two-dimensional lattices, $\tau = 2.05$. In the intermediate range $(2 < \delta < 4)$, the values of τ change with δ, and new universality classes emerge due to the competition between the spatial constraints and the long-range connections.

In summary, we have examined the percolation transition of spatially constrained Erdös-Rényi networks, where the distribution of the lengths of the edges follows a power law. We find that there are three distinct regimes where different types of percolation transitions with different critical exponents exist. For networks embedded in a square lattice, in the first regime $(\delta < 2)$, the giant component is characterized by nodes connected with long links comparable to the system size and the transition belongs to the universality class of percolation in ER networks. In the intermediate regime $(2 < \delta < 4)$, the giant component is comprised of localized cliques which are connected by few long links and the critical exponents seem to change with δ in a continuous way. Finally, in the last regime $(\delta > 4)$ the giant component has only short-range connections and the transition belongs to the universality class of percolation in two dimensions. For networks embedded in one dimensional chains, also three regimes occur: For $\delta < 1$, the spatial constraints are not relevant and the percolation properties are those of ER networks. For $\delta > 2$, there is no percolation transition as in linear chains. In the intermediate regime $(1 < \delta < 2)$, new percolation properties are observed. Analogous effects have been found for long-range links on fractal networks by Rozenfeld *et al.* [36]. Finally, we like to note that although our analysis has been performed on the square lattice, for reasons of universality we expect that the results will not change for different two-dimensional lattice or continuum structures. Moreover, we expect that similar three regimes will also appear for percolation in ER networks embedded in three dimensions.

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