# Propinquity drives the emergence of network structure and density

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Contributed by H. Eugene Stanley, August 2, 2019 (sent for review February 20, 2019; reviewed by Gyorgy Korniss and Dashun Wang)

**The lack of large-scale, continuously evolving empirical data usually limits the study of networks to the analysis of snapshots in time. This approach has been used for verification of network evolution mechanisms, such as preferential attachment. However, these studies are mostly restricted to the analysis of the first links established by a new node in the network and typically ignore connections made after each node's initial introduction. Here, we show that the subsequent actions of individuals, such as their second network link, are not random and can be decoupled from the mechanism behind the first network link. We show that this feature has strong influence on the network topology. Moreover, snapshots in time can now provide information on the mechanism used to establish the second connection. We interpret these empirical results by introducing the "propinquity model," in which we control and vary the distance of the second link established by a new node and find that this can lead to networks with tunable density scaling, as found in real networks. Our work shows that sociologically meaningful mechanisms are influencing network evolution and provides indications of the importance of measuring the distance between successive connections.**

network generation methods | network density | network evolution

The explosion in network research has been largely driven by the availability of big social data, by the analysis of social systems, and by studying the mechanisms behind the emergence of behavioral networks (1–10). Network generation methods are central in modeling network evolution and have helped us understand many properties of these systems, even when only a static snapshot is available. A large variety of mechanisms exist which have been proposed and verified (11), such as the famous preferential attachment principle (12), where nodes connect with higher probability to higher connected nodes. Different requirements may be imposed, such as requiring an unbiased configuration (13), and the mechanisms are usually adapted to the empirical systems that they attempt to explain.

In a typical network evolution model, new nodes are introduced into the system and they become connected to existing nodes according to certain rules. It is also possible that further changes can take place in the network, such as redirection of existing links, introduction of new links among existing nodes, etc. Recently, for example, Redner and coworkers (14, 15) studied a copying model, which is based on duplication-divergence mechanisms (16), and showed that a new node that inherits a fraction of connections from its first link can give rise to a diversity of topologies, mainly in terms of network density.

In the majority of these models, the rules for attaching a node specifically target the identification of the first connection. When a new node creates more than 1 connection, then the same rules are typically applied to identify each one of those connections; e.g., a node connects to $m$ nodes via preferential attachment (12). However, in a real evolving system the agents continue adding links for a long time after they are introduced in the network and it is highly unlikely that the processes of initial introduction are simply replicated over the complete lifespan of a node. This process of adding additional links is probably too complicated to observe in real networks or to model accurately. However, there is a tractable important question about the distance between the first $m$ connections of a new node which has not been explicitly addressed, even though it may be a key factor in defining central network properties, such as the network density.

Here, we present a first step that considers mechanisms that influence the choice of the second connection for newly introduced nodes. We suggest a model that can quite accurately capture the behavior of real-world evolution in empirical networks. The mechanism that we introduce here restricts the distance between the first and second connections of a new node, as measured prior to the node's introduction. As we show, the resulting network topology depends on the proximity of these 2 connections; we therefore call this the "propinquity model."

As a first demonstration that this metric can provide meaningful insight, we show that this distance does not behave trivially in empirical networks (Fig. 1). The network evolution in the 3 presented networks is known and we are therefore able to measure the distance between the first 2 connections for each new node just prior to the node's introduction. The resulting distance distribution cannot be characterized by a uniform distribution within the network; i.e., the distance of the second connection is not a randomly chosen quantity. On the contrary, each network seems to have its own characteristic distribution for these distances. In social networks, for example, shorter distances seem to be significantly preferred.

## Significance

**While many studies have focused on how new nodes make connections as they enter a network, we instead consider how choices of additional neighbors, after initial introduction, can shape patterns in emergent network structure. We find the footprints of this type of emergence in real-world networks and discuss how one could estimate the processes driving topology by examination of static snapshots of networks through the lens of link density. Our approach yields insight into network formation in applications ranging from social behavior to drug discovery.**

Author contributions: L.K.G., S.H., and N.H.F. designed research; L.K.G., S.H., and N.H.F. performed research; L.K.G., S.H., and N.H.F. analyzed data; and L.K.G., S.H., H.E.S., and N.H.F. wrote the paper.

Reviewers: G.K., Rensselaer Polytechnic Institute; and D.W., Northwestern University.

The authors declare no competing interest.



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This article contains supporting information online at www.pnas.org/lookup/suppl/doi:10.1073/pnas.1900219116/-/DCSupplemental.

First published September 23, 2019.

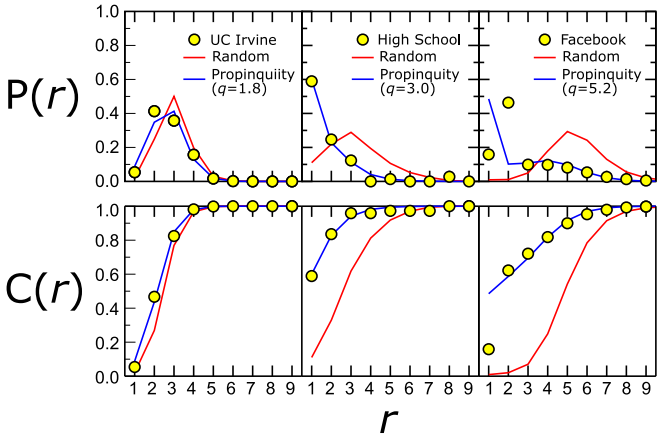


**Fig. 1.** Probability that the second link connects to a node at a distance $r$ from the first node. We measure the probability distribution (*Top*) and the cumulative distribution (*Bottom*) of the distance $r$ between the first 2 neighbors of a new node in a network. (*Left* to *Right*) Online social network in University of California, Irvine (17) ($N = 1,893$); high school friendship (18) ($N = 180$); and Facebook wall messages (19) ($N = 43,953$). Symbols represent the empirical results. The red lines correspond to the case where the second node would be selected in random ($q = 0$ in the propinquity model) (Fig. 2). The exponents $q$ in the propinquity model that give the best fit to the real data are shown in blue and represent the tendency of the distance to be smaller than random. Note that propinquity does not explain well the dynamics driving the Facebook wall messaging network for $r = 1$ and $r = 2$, but works well for larger $r$. The origin for this could be that initiation into a wall message network may be impacted more strongly by influence external to the online network (i.e., alternative means of communicating with friends and need for communication with friends of friends, apart from Facebook wall messages) and thus slightly skewing the results. Note, however, that the total probability for $r \leq 2$ is still consistent with the model prediction. Propinquity nevertheless offers meaningful and valuable insight as $r$ increases.

Using the underlying concept of our propinquity model, in this paper we explain the observed distance distributions in Fig. 1 and use this insight to propose proximity as a metric for characterizing the ongoing social dynamics of evolving networks in meaningful behavioral ways. We show how this characterization can lead to a systematic variation of network density, and we can use this metric to distinguish between network structures even when quantities such as the degree distribution and clustering coefficient seem identical.

## Model Description

**Local Network Density.** The underlying principle of network theory is that link structure among nodes provides more information than could be learned by examination of the nodes in isolation (20). In other words, connectivity is the main factor that determines the network behavior and response. Typical methods used to estimate the organization of links include, e.g., modularity or community detection (21), fractal properties (22), transport properties (23), percolation properties (24–26), etc. Surprisingly, little work has been done on direct measurements of link density in real networks [e.g., the concept of $n$-tangle density (27)]. However, the above approaches are mainly descriptive rather than predictive and there is currently no generic framework to detect potential mechanisms that explain the variation of local densities, especially at different system scales.

In terms of characterizing emergent density properties, there are 2 main families of growing network models. The most common mechanisms add a constant number of links for each node and, as a result, the link density is also constant, easy to calculate, and rarely given any further consideration [this is, e.g., the case of the preferential attachment mechanism (12)]. The second family of models uses a probabilistic mechanism of adding new links and can lead to either sparse or dense networks, depending on the model parameters [such as duplication-divergence models (28)]. In contrast to these 2 general cases, the propinquity model leads to networks that have a known global density, but (in contrast to earlier models) simultaneously enables a systematic variation of local density at different scales, as observed in real networks.

By focusing on the time-ordered behavior of local links and the resulting local density, and how this varies at different scales within the network, we can explain the emergence of communities and understand differences in the types of social dynamics that we observe in real-world networks. To quantify this local link density, scale is determined by the number of nodes, $n$, in a connected subgraph of the network. Formally, the link density $\rho$ in a graph with $N$ nodes and $L$ links is defined as the fraction of the number of links over the maximum possible number of links (29); i.e., $\rho = L/[N(N-1)/2]$. To measure the local link density we consider an induced connected subgraph of $n$ nodes, where we take into account all of the $e_n$ existing links between all pairs of nodes in the subgraph. We then define the local link density as

$$\rho_n = \frac{e_n}{\frac{n(n-1)}{2}}. \quad [1]$$

This allows us to study scaling of local link density as we vary the size of the subgraph, $n$. As explained in detail in *SI Appendix*, the behavior of this quantity is highly influenced by a trivial property. This is because we restrict ourselves to connected subgraphs of size $n$, which by definition requires all of the subgraphs to have at least $n-1$ links for connectivity. The simple solution that has been suggested is to subtract $n-1$ links from the numerator in Eq. **1** (27). In real networks the density has been shown to scale inverse linearly with the network size; i.e., $\rho_n \sim n^{-1} + O(n^{-2})$ (30, 31). This means that $e_n \sim n + O(1 + n^{-1})$ and the linear term dominates the behavior of $e_n$. For sparse networks where the prefactor of $n$ is close to 1, if we simply subtract these links from $e_n$, the density behavior will now depend on the higher-order terms, which may scale in a different way than $\rho_n$. We therefore apply here the recently defined metric (27) for the local $n$-tangle (topological analysis of network subgraph link/edge) density, $t_n$, as

$$t_n = \frac{e_n - (n-1)}{\frac{n(n-1)}{2} - (n-1)}. \quad [2]$$

The key feature in this definition is the removal of the $n-1$ links that are necessarily present in an induced subgraph to ensure connectivity. We also remove the same number of links in the denominator, so that $t_n$ remains properly normalized and ranges from $t_n = 0$ in the case of a loopless tree subgraph to $t_n = 1$ for a fully connected subgraph.

**The Predictive Power of the Propinquity Model.** For the model to be useful as a predictive tool, we must be able to validate hypotheses about the ways in which new nodes choose to connect to the network by agreement with observations of real-world network structures. There are already a large variety of network-growing models in the literature (12, 32–35). Typically, starting from a seed network, new nodes are introduced and attach themselves according to certain rules, e.g., by connecting preferentially to the most connected nodes. However, in many real systems nodes have a restricted freedom or ability to reach all of the available connections (36); thus the ability to create meaningful, behavioral-hypothesis–driven growing models would vastly expand our toolkit for understanding the mechanisms of ongoing social dynamics.

To model the varying strength of preference as a function of the network distance, we start with a small seed network of,

e.g., $N_0$ nodes connected to each other (indeed, any possible configuration of a connected network does not influence the results). The network grows by the introduction of a new node $i$ at each time step, when $i$ creates $m$ links toward the existing network. The first link is created randomly by choosing a node $j$ (either preferentially or uniformly). The new node $i$ then creates $m-1$ additional links, where a node $h$ is now selected to be connected to node $i$ with probability $r_{jh}^{-q}$. The distance $r_{jh}$ denotes the shortest distance in the existing network between nodes $j$ and $h$, and $q$ is a parameter that controls how close the new connections will remain to the first choice. A schematic description of the algorithm is shown in Fig. 2*A*. In Fig. 2 *B–D* we present some typical structures resulting from this algorithm for $m=2$, as we vary the value of $q$. The random character of the network at $q=0$ starts to break as we increase $q$, and fewer large-scale loops remain. For large values of $q$ the new nodes attach only to neighboring nodes, and the linear character of the network is preserved, with no long-range loops (Fig. 2*D*).

This model can describe a number of realistic situations. For example, new members that are invited into a social network will most likely connect to the close neighborhood of the member who invited them, and in spatially embedded networks, cost optimization makes shorter links preferable. Similarly, in copurchase networks, if 2 items are frequently bought together, there is a larger probability that a buyer will prefer a new item in the same category (37), which will remain within the extended neighborhood of these items. In this way, we assume that new connections favor to remain close to already existing connections of the same node (hence, "propinquity"). Even beyond the realms of association as individual choice, biological networks result from the gradual accrual of small mutations that alter functional pathways 1 change at a time. Altering the viability of an organism 1 mutation at a time can similarly be considered as a propinquity-driven process with the potential to explain dynamics of conserved complexes (38) and offer foundational frameworks for consideration of such network behaviors for applications including developmental biology (39) and drug discovery (40).

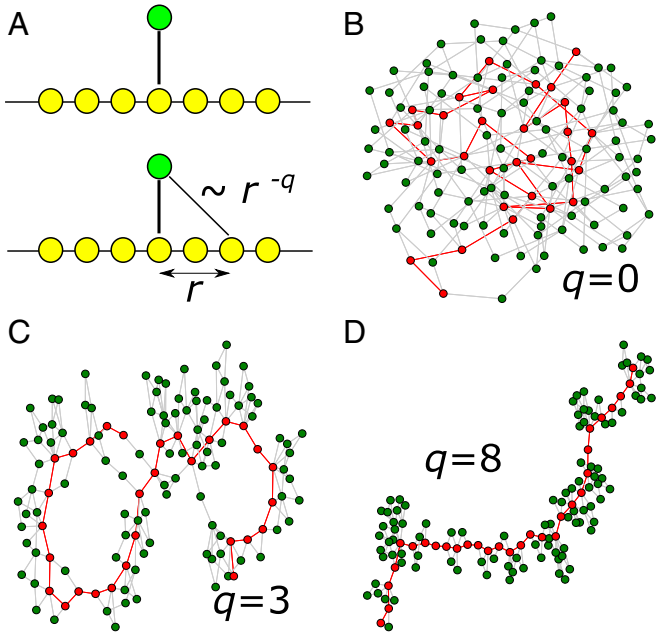


**Fig. 2.** The propinquity model. (*A*) The propinquity model can create networks with varying link density at different scales. The network grows via the successive addition of nodes (green) which link to a randomly selected existing node (thick line). The green node then selects a new node with probability $r^{-q}$, where $r$ is the distance from the previously selected neighbor. The network topology is controlled by varying the value of the parameter $q$. (*B–D*) Examples of small networks ($N=130$) created by varying the parameter $q$ in the propinquity model. The seed network includes 30 nodes in a line, which are shown in red, and 100 nodes are added, shown in green, according to the propinquity strategy with $m=2$ links. The structural differences are evident as we increase $q$ from $q=0$ (random recursive network) to larger values, such as $q=8$ (where new nodes remain locally connected and always form a triangle with 2 existing neighbors).

The limiting cases $q=0$ and $q=\infty$ correspond to random selections over the entire network and strictly neighboring selections, respectively. As $q$ increases we expect that the model will result in an increasingly modular structure, since the links remain local and there are very few links that connect distant parts of the network. At the same time, the value of $q$ controls the local density scaling, with direct impact on network topology.

## Results

**Results of the Model.** We have studied 2 main variants of the model, which differ in the attachment mechanism of the first connection. In the first variant, a new node selects its first connection randomly, while in the second variant, the selection is preferential, i.e., proportional to the degree of an existing node. It is quite straightforward to calculate the degree distribution for the limiting cases of both variants (*SI Appendix*). For random attachment, the distribution of the degree $k$ goes from exponential at $q=0$, $P(k)\sim(1+1/m)^{-k}$, to a power-law distribution $P(k)\sim k^{-\lambda}$ with an exponent $\lambda=2m+1$; i.e., $P(k)\sim k^{-(2m+1)}$, for large $q$. For preferential attachment, the degree distribution remains a power law with an exponent changing from $\lambda=2m+1$ at $q=0$ to an exponent $\lambda=3$ at large values of $q$ [where the propinquity model becomes similar to a growing Barabasi–Albert model (12)]. Note that for $m=1$ the exponent is $\lambda=3$; i.e., the propinquity model generalizes the BA network generation method. Critically, even though the 2 variants (random first selection with $q=8$ and preferential first selection with $q=0$) lead to the exact same degree distribution, they are structurally different. In the first case, we select a random node and the second selection connects to a neighbor of the first node, which leads to an effective preferential attachment mechanism for the second choice, where the network evolves by forming new triangles leading to a large clustering coefficient. In the second case, the first node is selected preferentially and the second node is selected randomly, so that the number of triangles (and therefore the clustering coefficient) is practically 0. In this example, the global link density and the degree distribution are identical, so the clustering coefficient can be used to separate these 2 cases. However, the clustering coefficient counts only loops of 3 nodes and by varying $q$ we can find examples where loops of larger sizes are favored over triangles, while the clustering coefficient is still very close to 0. The networks in this case seem statistically similar under most of the standard network measures, masking their fundamental differences in local density.

In the current study, we calculate the dependence of $\rho_n$ and $t_n$ (Eqs. **1** and **2**) on the sample size, $n$, by randomly sampling different parts of the network and averaging over the samples (see *SI Appendix* for details). We studied the possible scaling of $\langle t_n\rangle$ vs. $n$ and found that, typically, we recover a power-law behavior. This power-law form is described by the value of the exponent, $x$, in

$$\langle t_n\rangle \sim An^{-x}. \quad [3]$$

This scaling is more prominent for smaller values of $n$, when the subgraph size is significantly smaller than the network size, $N$. This is since our approach, due to the attractive interaction between successive links, is sensitive to local topologies where $n \ll N$. As we increase $n$, there is a crossover point after which $\langle t_n\rangle$ decays much faster with $n$, typically as $\langle t_n\rangle \sim n^{-1}$.

This approximate pattern is true for most cases that we studied, but the exact behavior of $\langle t_n \rangle$ can vary, depending on the structure.

Eq. **3** describes how the density of links changes as we increase the scale of observation, through the value of the exponent $x$. If $x$ is close to 0, this means that the $n$-tangle density remains constant at any size, while for larger values of $x$ the density decays faster, suggesting that larger areas of the network tend to become more tree-like. The variation of the exponent makes it also possible to monitor a possible transition of the structure in a given scale, from a tree to a denser graph, or vice versa. Note that the magnitude of the density is controlled by the value of the prefactor, $A$, independently of the scaling with the size.

The calculation of the exponent $x$ is straightforward for simple structures, such as Erdos–Renyi (ER) networks and lattices (*SI Appendix*). In ER networks, there is no variation of the density with $n$, so that $x = 0$. In lattices, as we discuss in the next section, the asymptotic value of the exponent is $x = 1$. In general, the exponent $x$ can vary between 0 and 1, and therefore the lattice and the random network are representative of 2 extreme behaviors of how density can scale with size. Clearly, this means that we can characterize networks in this way as being closer to, or farther from, particular structures, such as in the case of lattice or random networks (28). Note that using the standard definition of local density $\rho_n$ in Eq. **1**, we always retrieve the trivial behavior $\rho_n \sim n^{-1}$, which does not carry any useful information on local density.

We used the case of $m = 2$ links per new node. As expected, when $q = 0$, the connections are all random and we recover the result for random ER networks, where $t_n$ does not change significantly with $n$. As we increase the value of $q$ the density starts to change systematically with $n$, following a power-law behavior (Fig. 3*A*). This is reflected in the value of the exponent $x$ which starts at $x = 0$ when $q = 0$ and increases monotonically until it reaches values close to $x \sim 1$ (Fig. 3*B*).

Interestingly, while the local density changes drastically with $q$, and we can therefore deduce that large structural changes take place, we would not be able to observe these changes by using standard network measures, such as clustering and distances. In Fig. 3*C*, the clustering coefficient remains almost 0 for values between $q = 0$ and $q \sim 4$, but the local density behavior is drastically different, as can be seen in the results of Fig. 3*A* and the slope calculations in Fig. 3*B*. Similarly, the network diameter remains unchanged in the range of $q$ from 0 to 4 (Fig. 3*D*). In the same $q$ range, the slope of the density increases from 0 to 0.6. These results show that even though the relative distances remain constant, the links reorganize themselves in a systematic way with larger local densities at small subgraphs. The local link density exponent can therefore be used to characterize changes in network structures that cannot be predicted by the study of the clustering coefficient or shortest paths. When $q$ assumes large values, both the clustering coefficient and the network diameter increase significantly as a result of highly localized connections and the removal of practically all network shortcuts. However, in this range there is very little variation in the local density, $t_n$ (Fig. 3*B*).

**Real Networks.** In real systems, when a node creates a new link, there are obviously many possible mechanisms in action, e.g., homophily and collective action (41), consensus dynamics (42), etc. The propinquity model, however, allows us to isolate the influence of the neighbor's proximity to network density. It then provides a simple model by which to predict the variation of link density at different scales, even though the use of the typical link density definition would falsely indicate that the extent of the propinquity concept (through the parameter $q$) should have no influence on the results.

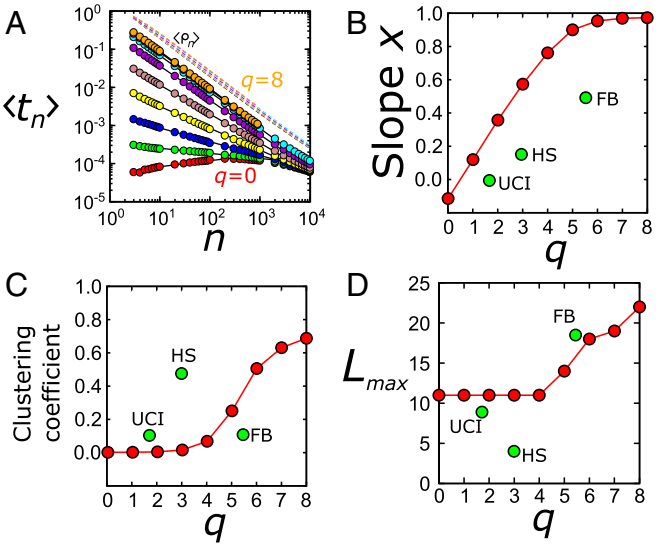


**Fig. 3.** Results for the propinquity model. Here, the first link of a new node attaches preferentially to the existing network. (*A*) Scaling of *n*-tangle density as a function of *n*. From bottom to top, the value of *q* increases from 0 to 8 in steps of 1. Dashed lines correspond to regular density $\langle \rho_n \rangle$, where there is no observable effect of *q* (the slope remains constant). (*B*) Calculation of the exponents *x* for the lines in *B*, as a function of *q*. The green circles indicate the corresponding values for the empirical networks analyzed in Fig. 1. (*C*) Clustering coefficient as a function of *q*. (*D*) Even though the exponent *x* increases with *q*, the network diameter (as well as the clustering) in the propinquity model remains constant up to $q = 4$ and increases only for larger values of *q*.

In Fig. 4 we calculate the $n$-tangle density scaling for the 3 empirical networks analyzed in Fig. 1. Each network leads to a different slope, $x$. Using the optimal value for $q$ from Fig. 1 and the exponent $x$ from Fig. 4, we can compare the propinquity metrics for these networks. Of course, as mentioned above, the empirical data cannot be assumed to be fully described by 1 mechanism alone. However, it is clear from Fig. 3*B* that there is a consistent trend in both the model and empirical data that larger local density variations appear at larger $q$ values. This observation is important because it provides a link between the analysis of a static network snapshot and the network generation mechanism, which is difficult to observe directly. In practice, we have shown that measurements of the scaling of local link density provide a systematic way to understand network growth mechanisms which are based on the distance between 2 nodes, added one after another as friends.

As a comparison, network properties such as the clustering coefficient or the network diameter (shown in Fig. 3 *C* and *D*) do not suggest any clear trends with $q$. However, this may also be attributed to the small size of these networks, such as the high school network, which contains only 180 students and is an unusual, dense network.

## Discussion

Our work demonstrates the importance of incorporating mechanisms of attachment that allow the tailoring of local network densities to achieve realistic network structures in generative growing models. We have studied the simple case where the second link depends on the in-network distance and we have shown that this leads to very different topologies. This finding was confirmed by studying the distance between the first 2 neighbors of new nodes in empirical networks.

We establish a family of network generation models where the subsequent connections depend on the distance between 2 nodes. To detect the influence of this mechanism on topology we

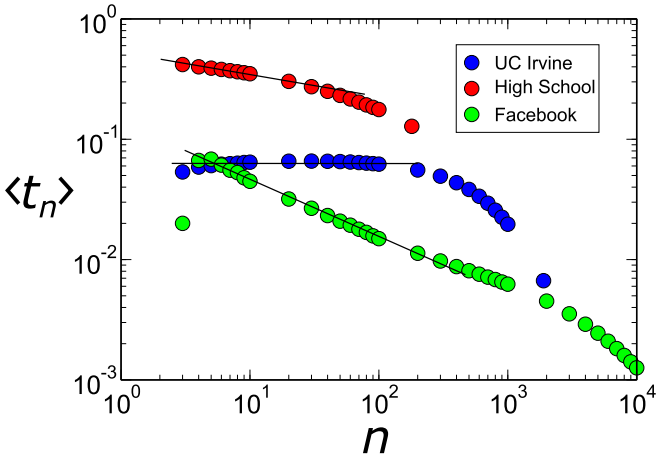


**Fig. 4.** Density scaling of the real networks in Fig. 1. The exponent, $x$, that characterizes the density scaling in the empirical networks of Fig. 1 is consistent with the propinquity model exponents $q$. These exponents ($x$ = 0, 0.16, and 0.5) are shown in Fig. 3*C* and follow the same trend, increasing with $q$, as the propinquity model in that plot.

study the scaling of local density. If we use the standard definition of local density, then the scaling is dominated by a trivial structure. However, we show that a redefinition of local density, Eq. **2**, provides a direct way of studying this scaling and the local density can probe the structure at different scales.

From a theoretical point of view, the power-law behavior in Eq. **3** can also be seen as the definition of an additional fractal dimension for complex networks, albeit within the range from $x = 0$ to $x = 1$. The traditional definition of a fractal object detects how the mass scales with distance. In complex networks, this definition becomes problematic because of the natural restriction of distances in usually just 1 decade. For example, the maximum distance in the 3 empirical networks used in Fig. 1 ranges from 4 to 19, which does not allow a reliable evaluation of network dimensions (see also related discussion in *SI Appendix*). There are many methods in the literature that have introduced possible modifications on how fractal features can be measured in networks (43, 44), but even then there are many nonfractal networks (e.g., ER networks) whose structural differences cannot be captured by fractal dimension. As an alternative to these methods, the present link density method can provide a natural interpretation of the self-similar properties of a network. In this definition, the important quantity is the "mass" of the links instead of the number of nodes (12), while the "length" corresponds to the number of nodes, instead of a distance metric. Self-similarity in this study shows how the fraction of the excessive links scales with the number of nodes. A small exponent means that any part of the network will have similar link density, independently of the sampled size, but a large exponent shows that larger samples of the network become sparser. The rate at which the density decreases is then determined by this fractal exponent $x$.

In conclusion, the propinquity model provides another class of generative models, rooted in features of real networks, and is leading us to understanding how individuals become integrated into communities at different scales. It enables us to test meaningful hypotheses about which scales of social interactions are important in an evolving network as a metric for isolated analysis and comparison between systems. Most importantly, it allows us to make behaviorally driven predictions about the emergent structure of networks based on single snapshot observations.

**ACKNOWLEDGMENTS.** We acknowledge support by the National Science Foundation (NSF) under Grants CNS-1646856 (to L.K.G.) and CNS-1646890 (to N.H.F.). S.H. acknowledges financial support from US–Israel Binational Science Foundation (NSF–BSF) No. 2015781; Israel Science Foundation (ISF); Office of Naval Research (ONR) Grant N62909-14-1-N019; the Israel Ministry of Science and Technology with the Italian Ministry of Foreign Affairs; the Israel Ministry of Foreign Affairs with the Japan Science Foundation; and the Army Research Office. H.E.S. and S.H. acknowledge financial support from Defense Threat Reduction Agency (DTRA) Grant HDTRA-1-14-1-0017.

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