Localization of fractons, random walks and linear polymers in percolation

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Abstract

We review analytical and numerical results for the vibrational amplitudes of localized excitations, the probability distribution of random walks and the distribution of linear polymers (modeled by self-avoiding walks of N steps) on percolation structures at criticality. Our numerical results show that the fluctuations of these quantities, at fixed shortest-path distance ("chemical length") ℓ from the center of localization, are considerably smaller than at fixed Euclidean distance r from the center. Using this fact, we derive via convolutional integrals explicit expressions for the averaged functions in r-space, and show analytically and numerically that three different localization regimes occur. Remarkably is that in the short-distance regime, the averages show a universal spatial decay behavior, with the same exponent for both fractons and random walks, while in the asymptotic regime, the averages depend explicitely on the number of configurations considered.

1. Introduction

It is well known that in random fractal structures vibrational excitations can be localized, i.e. the mean vibrational amplitudes of the vibrational excitations, called fractons, decay with increasing distance r from a localization center [1-9]. The knowledge of the localization behavior on fractals is relevant for a large number of both experimental and theoretical issues [10-14], e.g. inelastic scattering [10-13] in disordered systems that are self similar on certain length scales. Closely related to the localization behavior of fractons [1,2,7-9,15] is the way how the mean probability densities of random walks and self-avoiding walks on random fractal structures decrease with increasing distance r from the starting point [1,2,13,21]. The probability density of a random walker is important for all physical processes that involve diffusing particles, e.g., in chemical reactions between diffusing reactants [15], while self-avoiding walks (SAW) [22] are generally accepted for modelling linear polymers in good solvents. Most of the work on linear polymers in disordered media has been concentrated so far on the problem how the mean end-to-end distance of a polymer chain depends on the number of monomers N of the polymer [23,24]. ¿From a microscopic point of view more important is the configurational averaged distribution function, which is defined as the probability that a polymer of N monomers has an end-to-end distance r. The mean end-to-end distance is only the first moment of the distribution and contains only part of the information.

In this paper, we review the results of the vibrational excitations and of the probability distributions of random walks and self avoiding walks on random fractal structures we obtained recently [4,5,19,21]. We find that the *asymptotic* regimes of the relevant localization functions describing the three localization phenomena are divided into two parts [25,27]. There exists a crossover distance $r_\times(N_{\rm av})$ that increases logarithmically with the number $N_{\rm av}$ of configurations taken into account in the averages $\langle\rangle$. In general, the localization functions decay, for large distances r from the center of localization, according to $\exp[-\text{const } r^{d_\Psi}]$. The localization exponent d_Ψ differs below and above $r_\times(N_{\rm av})$. Below $r_\times(N_{\rm av})$, we have $d_\Psi = d_w/(d_w-1)$ (random walks), $d_\Psi = 1$ (fractons) and $d_\Psi = 1/(1-\nu_r)$ (polymers), while above $r_\times(N_{\rm av})$, $d_\Psi = d_{\rm min} d_w/(d_w - d_{\rm min})$ (random walks), $d_\Psi = d_{\rm min}$ (fractons) and $d_\Psi = d_{\rm min}/(1-\nu_r d_{\rm min})$ (polymers). Here, d_w describes how the root-mean-square displacement $\langle R(t)\rangle \sim t^{1/d_w}$ of a random walker

scales with time t and $d_{\min}$ is the fractal dimension of the shortest path between two points at Euclidean distance r on the fractal, $\langle \ell(r) \rangle \sim r^{d_{\min}}$. For SAW's of N steps, the exponent ν_r describes how the mean end-to-end distance $\langle r(N) \rangle$ of the linear chains scales with N, $\langle r(N) \rangle \sim N^{\nu_r}$.

In the short-distance regime, random walks and fractons behave the same and decay with the same exponent g, which is a purely structural quantity, describing the distribution of the minimum path distance ℓ between two points at fixed Euclidean distance r [26], (see also [30]). Since random walks and vibrational excitations require a slightly different treatment than polymer chains, we divide this paper into 2 parts. We start with a discussion of fractons and random walks.

2. Fractons and Random Walks

To treat both localization phenomena simultaneously, we introduce a function $\psi_i(r)$, which stands either for the amplitude $\mid u_i(r,\omega) \mid$ of the displacement of a vibrating particle on a site i at distance r from the localization center, or for the probability $p_i(r,t)$ to find a random walker (after t time steps) on a site i at distance r from its starting point on a given configuration of the random fractal substrate.

Our basic assumption is that the shortest-path distance ("chemical length") ℓ rather than the Euclidean distance r between two points on the fractal is the relevant physical length in the problem, such that the fluctuations of the $\psi_i(\ell)$ on sites i at fixed chemical distance ℓ from the localization center are small compared with the fluctuations of the $\psi_i(r)$ at fixed Euclidean distance r from the localization center, for both the same and different configurations. According to [5,19], we can approximately neglect the fluctuations in ℓ-space and write

$$\psi_i(\ell) \cong \Psi(\ell) \sim \exp[-(\ell/\xi_\ell)^v]. \tag{1}$$

For fractons, we have $\xi_\ell \equiv \xi_\ell(\omega) \sim \omega^{-d_w/2d_{\min}}$ and $v = 1$. For random walks, ξ_ℓ is proportional to the mean chemical distance $\langle \ell(t) \rangle \sim t^{d_{\min}/d_w}$ and $v = d_w/(d_w - d_{\min})$. Assumption (1) is trivial for random walks on linear fractal structures ($v = 2$) or for localized vibrations on a linear fractal with a small impurity mass located at the origin ($v = 1$). Figure 1 illustrates that the assumption is also reasonable for random walks on percolation structures. In the figure we show the distribution of the $p_i(\ell,t)$ and $p_i(r,t)$, for fixed time $t = 1000$, $\ell = 150$ and $r = 70$, averaged over $5 \cdot 10^3$ configurations. The

figure shows clearly that the distribution of the p_i for fixed ℓ is narrow compared with the logarithmically broad distribution of the p_i for fixed r. Indeed, from assumption (1) one can derive the functional form of the histogram for the p_i in r-space (full line in the figure), which fully coincides with the result from the computer simulations [15]. Qualitatively similar results were obtained for the fracton amplitudes in percolation structures [5].

To obtain the mean localization functions, one first averages $\psi_i(r)$ over all N_r sites i at distance r from the localization center for a single configuration of the fractal substrate, according to [15]

$$\Psi(r) = \frac{1}{N_r} \sum_{i=1}^{N_r} \psi_i(r). \tag{2}$$

Assuming that among the N_r sites $N_\ell(r)$ sites are at chemical distance ℓ from the center and employing Eq. (1), we can write [25]

$$\Psi(r) = \frac{1}{N_r} \sum_{\ell=\ell_{\min}(r)} N_\ell(r)\Psi(\ell)$$

where $\ell_{\min}(r)$ is the shortest chemical length on the fractal reaching the distance r from the center. Averaging over $N_{\rm av}$ configurations and replacing the sum by an integral, yields the localization function of interest,

$$\langle \Psi(r) \rangle_{N_{\rm av}} = \int_{\ell_{\min}(r,N_{\rm av})}^{\infty} \phi(\ell \mid r; N_{\rm av}) \Psi(\ell) d\ell \ , \tag{3}$$

where $\phi(\ell \mid r; N_{\rm av})$ is the average of $N_\ell(r)/N_r$ over $N_{\rm av}$ configurations for a range $d\ell$ near ℓ, $\ell_{\min}(r, N_{\rm av})$ is defined as the shortest chemical length on the fractal, among the $N_{\rm av}$ configurations, reaching the distance r from the center, and $\phi(\ell \mid r; N_{\rm av}) = 0$ for $\ell < \ell_{\min}(r, N_{\rm av})$.

Equation (3) reveals that averaging the localization function involves both a simple arithmetic average ($\phi(\ell \mid r; N_{\rm av})$) that is independent of $N_{\rm av}$ for large $N_{\rm av}$ and a minimization procedure ($\ell_{\min}(r, N_{\rm av})$). Naturally, the minimum value $\ell_{\min}(r, N_{\rm av})$ depends strongly on $N_{\rm av}$, and this causes, as we shall see below, the breakdown of self-averaging and scaling at large distances r. The average $\phi(\ell \mid r; N_{\rm av})$ can be approximately written as [4] (see also [1,15,21,28,29])

$$\phi(\ell \mid r; N_{\rm av}) \equiv \phi(\ell \mid r) = \frac{C_1}{\ell} \left(\frac{r}{\ell^{1/d_{\min}}}\right)^g \exp\left[-C_2 \left(\frac{r}{\ell^{1/d_{\min}}}\right)^{\tilde{\delta}}\right] \tag{4}$$

with $\tilde{\delta} = d_{\min}/(d_{\min}-1)$. For percolation clusters, where $d_{\min} \cong 1.13$ in $d=2$ and $d_{\min} \cong 1.37$ in $d=3$ [15], one has $g \cong 1.35$ in $d=2$ and $g \cong 1.5$ in $d=3$ [21], for not too small $r/\ell^{1/d_{\min}}$. For asymptotically small $r/\ell^{1/d_{\min}} \ll 1$, the values for g are slightly lower [29]. For fixed r, $\phi(\ell \mid r)$ has its maximum at $\ell_{\max}(r) \cong r^{d_{\min}}$. As shown in Fig. 2a, the cut-off value $\ell_{\min}(r, N_{\rm av})$ decreases monotonically with $N_{\rm av}$ until the absolute minimum $\ell_{\min} \equiv r$ is reached, according to [25]

$$\ell_{\min}(r, N_{\rm av}) = \begin{cases} r, & r \ll r_c(N_{\rm av}), \\ \alpha_{\min}(N_{\rm av}) r^{d_{\min}}, & r \gg r_c(N_{\rm av}). \end{cases} \tag{5}$$

The crossover value $r_c(N_{\rm av})$ and the prefactor $\alpha_{\min}(N_{\rm av})$ can be determined analytically: since the probability $W(N_{\rm av})$ of finding a percolation structure at the critical concentration p_c with $\ell = r = r_c$ in a square- or s.c. lattice with coordination number z is $W(N_{\rm av}) = 1/N_{\rm av} = z \cdot p_c^{r_c}$, we have

$$r_c(N_{\rm av}) = (\ln z + \ln N_{\rm av})/\ln(1/p_c). \tag{6}$$

The unknown prefactor $\alpha_{\min}(N_{\rm av})$ in (5) can be determined from the scaling assumption, $\ell_{\min}(r, N_{\rm av}) = r_c g(r/r_c)$. In order to satisfy (5), we must require $g(x) = x$ for $x \ll 1$ and $g(x) = g_1 x^{d_{\min}}$ for $x \gg 1$. This yields

$$\alpha_{\min}(N_{\rm av}) = g_1 r_c^{1-d_{\min}}. \tag{7}$$

Figure 2b shows $\ell_{\min}/r_c$ versus r/r_c for the same $N_{\rm av}$ values as in Fig. 2a. The data collapse supports strongly the scaling ansatz, and shows that $g_1 \cong 1$ for percolation.

Using (4-7), the integral (3) can be calculated analytically. For $\ell_{\min}(r, N_{\rm av}) < \ell < \ell_{\max}(r)$, the integrand in (3) is the product of two exponential functions $\Psi \cdot \phi \sim \exp[-C_2(r/\ell^{1/d_{\min}})^{\tilde{\delta}} - (\ell/\xi_\ell)^v] \equiv \exp[-\eta(\ell)]$, which shows a steep maximum at $\ell^* = \xi_r^{d_{\min}}(r/\xi_r)^{u/v}$, with $\xi_r^{d_{\min}} = \xi_\ell(C_2/(v(d_{\min}-1)))^{1/v}$ and $u = v d_{\min}/(1+v(d_{\min}-1))$. Applying the method of steepest descent, we obtain [5,19]

$$\ln\langle\Psi(r)\rangle_{N_{\rm av}} \sim -\eta(\ell^*) \sim -\left(\frac{r}{\xi_r}\right)^u. \tag{8}$$

For fractons, $u=1$ and ξ_r is proportional to the standard localization length $\lambda(\omega)$. For random walks, $u = d_w/(d_w-1)$ and ξ_r is proportional to $\langle R(t)\rangle$.

By definition, (8) holds only for $\ell_{\min}(r, N_{\rm av}) < \ell^* < \ell_{\max}(r)$, and this restriction determines the r-regime $r_1 < r < r_\times(N_{\rm av})$ where (8) is valid. We find

$$r_1 = \xi_r[(g + d_{\min})/C_2\tilde{\delta}] \tag{9}$$

$$r_\times(N_{\rm av}) \cong \xi_r r_c^{1/u}(N_{\rm av}). \tag{10}$$

For $r > r_\times(N_{\rm av})$, the integrand in (3) is peaked sharply at $\ell = \ell_{\rm min}(r, N_{\rm av})$, and

$$\ln\langle\Psi(r)\rangle_{N_{\rm av}} \sim -\left(\ \ell_{\rm min}(r, N_{\rm av})/\xi_\ell\ \right)^v \sim -r_c(N_{\rm av})^{v(1-d_{\rm min})}(r/\xi_r)^{v\,d_{\rm min}},$$

$$r > r_\times(N_{\rm av}). \tag{11}$$

Figure 3 shows our results for the asymptotic regime $r > r_1$ for site percolation clusters on the square lattice. Figure 3a compares the logarithm of the normalized mean probability density $(-\ln[\langle p(r,t)\rangle_{N_{\rm av}}/\langle p(0,t)\rangle_{N_{\rm av}}])$ of random walks for several $N_{\rm av}$-values with the theoretical predictions (straight lines). The $N_{\rm av}$-dependent crossover $r_\times(N_{\rm av})$ of the data is clearly seen and agrees with the theoretical predictions. The slopes of the lines correspond to our predictions $d_\psi = 1.53$, for $r < r_\times$ and $d_\psi = 1.86$, for $r > r_\times$. Figure 3b shows the logarithm of the normalized mean fracton amplitude $(-\ln\langle|u(r,\omega)|/|u(0,\omega)\rangle)$ as a function of $r/\xi_\ell^{1/d_{\rm min}}$, for the "typical" average introduced in [5] and corresponding to the case $N_{\rm av} = 1$ in (5-11) [25]. The localization length in ℓ-space $\xi_\ell \sim 29$ was also found in Ref. [5]. The crossover $r_\times(N_{\rm av} = 1)$ is expected to occur for $r_\times(1)//\xi_\ell^{1/d_{min}} \cong 8.7$ and separates the regime with $d_\psi = 1$, when $r < r_\times$, from the regime with $d_\psi = d_{\rm min}$, when $r > r_\times$. However, the latter regime $r > r_\times$ can not be reached by our present data.

Next we discuss the short-distance regime $r < r_1$ [26]. To determine the integral (3) in an efficient way we note that $r < r_1$ implies $\ell_{\rm max}(r) < \xi_\ell \simeq r_1^{d_{\rm min}}$. Hence for $\ell < \xi_\ell$, the behavior of $\phi(\ell \mid r)$ differs strongly from the behavior of $\Psi(\ell)$: while $\phi(\ell \mid r)$ shows a steep maximum at $\ell_{\rm max}$, $\Psi(\ell)$ is nearly constant. Hence we can assume that to a very good approximation, the integrand of (3) can be written as

$$\phi(\ell \mid r)\Psi(\ell) \cong \begin{cases} \phi(\ell \mid r)\langle\Psi(0)\rangle, & \ell \le \xi_0, \\ 0, & \ell > \xi_0. \end{cases} \tag{12}$$

The cutoff-length ξ_0 is determined by the condition

$$\int_0^\infty \Psi(\ell)^\tau\, \ell^{d_\ell - 1}\, d\ell = \int_0^{\xi_0} \langle\Psi(0)\rangle^\tau\, \ell^{d_\ell - 1}\, d\ell, \tag{13}$$

which for random walks ($\tau = 1$) is the normalization condition and for fractons ($\tau = 2$) reflects energy conservation. The exponent d_ℓ describes how the fractal mass scales

with ℓ, $\langle M(\ell)\rangle \sim \ell^{d_\ell}$, with $d_\ell \cong 1.678$ and $d_\ell \cong 1.885$ for percolation in $d = 2$ and $d = 3$, respectively [15]. Inserting (1) into (13) we obtain $\xi_0 = \Gamma(\frac{d_\ell}{v} + 1)^{1/d_\ell} \tau^{-1/v} \xi_\ell$. Substituting (12) into (3) yields [26]

$$\langle \Psi(r)\rangle \cong \langle \Psi(0)\rangle \int_0^{\xi_0} \phi(\ell \mid r) d\ell = \langle \Psi(0)\rangle \left(\int_0^{\infty} \phi(\ell \mid r) d\ell - \int_{\xi_0}^{\infty} \phi(\ell \mid r) d\ell \right). \qquad (14)$$

Since $\int_0^{\infty} \phi(\ell \mid r) d\ell = 1$ and $\phi(\ell \mid r) \cong (C_1/\ell)(r/\ell^{1/d_{\min}})^g$ for $\ell \gg \ell_{\max}$ we obtain finally in the short distance regime ($r < r_1$)

$$\frac{\langle \Psi(r)\rangle}{\langle \Psi(0)\rangle} = 1 - A \left(\frac{r}{\xi_\ell^{1/d_{\min}}} \right)^g, \qquad (15a)$$

with

$$A = \frac{C_1 d_{\min}}{g} \left(\frac{\tau^{1/v}}{\Gamma(\frac{d_\ell}{v} + 1)^{1/d_\ell}} \right)^{g/d_{\min}}. \qquad (15b)$$

According to (15), the decay of $\langle \Psi(r)\rangle$ is solely characterized by the substrate geometry represented by the structural exponent g in (4) [26]. This exponent depends explicitely on the dimension d of the embedding space. It can be shown, however, that (15a) is valid only below a critical dimension d_c, above which the exponent g is replaced by another exponent g_c which is independent of d. For random walks one finds $g_c = d_w$, with $d_c = 6$ on linear fractal structures generated by random walks and self-avoiding walks, and $d_c = 10$ on percolation clusters at criticality [30].

The results for the short-distance regime $r < r_1$ are plotted in Fig. 4, which shows (a) the mean probability density of random walks $\langle p(r,t)\rangle / \langle p(0,t)\rangle$ versus $(r/\xi_\ell^{1/d_{\min}})^g$ and (b) the mean amplitudes $\langle \mid u(r,\omega) \mid \rangle / \langle \mid u(0,\omega) \mid \rangle$ of localized vibrational excitations versus $(r/\xi_\ell^{1/d_{\min}})^g$, both for site percolation clusters on the square lattice. The figures show that our predictions are in full *quantitative* agreement with the numerical results. The full lines represent our theoretical results, Eq. (15), with no fit parameters involved. The values of ξ_ℓ have been determined numerically from the decay of the corresponding mean localization functions in ℓ-space.

3. Polymers in disordered media

Next we study the probability distributions of linear polymers modeled by N-step self-avoiding random walks in disordered structures, again both in r- and in ℓ-space. In our calculations, SAWs of N steps (N monomers) are generated by the enrichment method (for details see [27]), and as in the foregoing section, the disordered structure is modeled by the incipient percolation cluster on a square lattice.

Due to the self-avoiding constraint, however, SAWs cannot explore the hole percolation cluster but only the percolation backbone. Thus, in contrast to the definitions in Section 2, the distribution functions studied here are normalized on the percolation backbone instead of on the entire cluster. In ℓ-space, we evaluate the probability $P(\ell, N)$ that after N-steps the endpoints of the SAW are separated by the chemical distance ℓ. Correspondingly, the quantity $P(r, N)$ represents the probability that after N-steps the endpoints of the SAW are separated by the Euclidean distance r.

Since for SAWs, the relevant disordered substrate is the percolation backbone, we expect that the exponents describing the backbone rather than those describing the percolation structure will enter the probability distributions $P(\ell, N)$ and $P(r, N)$. In spite of these differences we find that, similarly to the random walk case, the fluctuations in ℓ-space are significantly smaller than the fluctuations in r-space and derive, in analogy to the treatment of random walks and fractons, an analytical expression for the asymptotic behavior of the average probability density in $r-$space, $\langle P(r, N)\rangle$.

First we discuss the mean chemical end-to-end distance of SAW's on $2d$ percolation clusters at criticality, $\langle \ell(N)\rangle$. Our numerical results (see Ref. [27]) suggest that $\langle \ell(N)\rangle$ follows a power law $\langle \ell(N)\rangle \sim N^{\nu_\ell}$ with $\nu_\ell = 0.87 \pm 0.02$. We note that for $d \geq 6$, it is expected that $\nu_\ell = 1$ [27]. Since $\ell \sim r^{d_{\min}}$, ν_ℓ can be related to the exponent ν_r describing the mean end-to-end distance in r-space, $\langle r(N)\rangle \sim N^{\nu_r}$ by $\nu_r = \nu_\ell / d_{\min}$.

Next we focus on the average probability $\langle P(\ell, N)\rangle$. Figure 5 shows $\ell\langle P(\ell, N)\rangle$ versus ℓ/N^{ν_ℓ} and suggests that for $\ell/\xi_\ell > 0.28$,

$$\langle P(\ell, N)\rangle \sim \ell^{-1}\ (\ell/\xi_\ell)^{g'_\ell}\ \exp[-(\ell/\xi_\ell)^{\delta_\ell}] \tag{16a}$$

with $\delta_\ell = 1/(1-\nu_\ell)$, $g'_\ell = 3.0 \pm 0.2$, and $\xi_\ell \cong 1.27 N^{\nu_\ell}$. For $\ell/\xi_\ell < 0.28$, the data suggest that

$$\langle P(\ell, N)\rangle \sim \ell^{-1}\ (\ell/\xi_\ell)^{g_\ell} \tag{16b}$$

with $g_\ell = 2.5 \pm 0.2$. Notice that $\langle P(\ell, N)\rangle$ is normalized according to $\int d\ell \langle P(\ell, N)\rangle = 1$.

We have recently shown that for self-avoiding walks in $2d$ percolation clusters at criticality, the fluctuations in ℓ-space are much smaller than in r-space [27]. According

to this result, we can proceed in a similar way as in section 2. We define the probability $p_i(r,N)$ that after N steps the SAW has arrived at a backbone site i at distance r from the origin of the SAW, on a given configuration. Assuming that there are N_r^B sites on the backbone at distance r in the considered configuration, the probability $P(r,N)$ for that configuration is now given by (cf. Eq. (2))

$$P(r,N) = \sum_{i=1}^{N_r^B} p_i(r,N). \tag{17}$$

Among the N_r^B sites, there are $N_\ell^B(r)$ sites at chemical distance ℓ from the center, and we can write

$$\begin{aligned} P(r,N) &= \sum_{\ell=\ell^{\rm B}_{\rm min}(r)}^{\infty} N_\ell^B(r) p_i(\ell,N) \\ &= \sum_{\ell=\ell^{\rm B}_{\rm min}(r)}^{\infty} \frac{N_\ell^B(r)}{N_\ell^B} P(\ell,N)\ . \end{aligned} \tag{18}$$

In deriving the second equality, we have assumed that there are N_ℓ^B sites at chemical distance ℓ from the center and that $p_i(\ell,N) \simeq P(\ell,N)/N_\ell^B$. This is justified, since the fluctuations of the $p_i(\ell,N)$ are small. By definition, $\ell^{\rm B}_{\rm min}(r)$ is the length of the shortest chemical path between the center and any site i on the backbone at fixed distance r in the considered configuration. Assuming further that, similarly to the random-walk case, the fluctuations of $P(\ell,N)$ between different configurations are small, we can write $P(\ell,N) \simeq \langle P(\ell,N)\rangle$. Averaging over $N_{\rm av}$ configurations and replacing the sum in (18) by an integral, yields the desired relation between $\langle P(\ell,N)\rangle$ and $\langle P(r,N)\rangle_{N_{\rm av}}$,

$$\langle P(r,N)\rangle_{N_{\rm av}} = \int_{\ell^{\rm B}_{\rm min}(r,N_{\rm av})}^{\infty} \phi^B(r \mid \ell; N_{\rm av})\ P(\ell,N)\ d\ell, \tag{19}$$

where $\phi^B(r \mid \ell; N_{\rm av}) \equiv \phi^B(r \mid \ell) \equiv N_\ell^B(r)/N_\ell^B$, is the probability that two sites on the backbone separated by chemical distance ℓ are at Euclidean distance r. We like to note that $\phi^B(r \mid \ell)$ is not fully analogous to $\phi(\ell \mid r)$ in Eq. (4). While $\phi^B(r \mid \ell)$ is normalized in r-space, $\int_0^\infty \phi^B(r \mid \ell) dr = 1$, $\phi(\ell \mid r)$ is normalized in ℓ-space, $\int_0^\infty \phi(\ell \mid r) d\ell = 1$. Recently, it has been shown that $\phi^B(r \mid \ell)$ can be written in a form analogous to Eq. (4) [31],

$$\phi^B(r \mid \ell) = \frac{C_1^{\rm B}}{r} \left(\frac{r}{\ell^{1/d_{\rm min}}}\right)^{g_B} \exp\left[-C_2^{\rm B}\left(\frac{r}{\ell^{1/d_{\rm min}}}\right)^{\tilde\delta}\right] \tag{20}$$

with $\tilde{\delta} = d_{\min}/(d_{\min}-1)$. For fixed r, $\phi^B(r \mid \ell)$ has its maximum at $\ell^B_{\max}(r) \cong r^{d_{\min}}$.

The lower integration limit in (19), $\ell^{\rm B}_{\min} = \ell^{\rm B}_{\min}(r, N_{\rm av})$, is that of the backbone, which is larger than $\ell_{\min}$ for the entire percolation cluster, but still obeys an equation similar to (5) with $\alpha_{\min}(N_{\rm av})$ replaced by $\alpha^{\rm B}_{\min}(N_{\rm av}) \sim r_c^{\rm B}(N_{\rm av})^{1-d_{\min}}$ and $r_c^B(N_{\rm av}) \sim \ln(N_{\rm av})$.

Equation (19) has the same functional form as Eq. (3) and can be solved in exactly the same way as described above for random walks and fractons. Following the procedure outlined in Sect. 2 (Eqs. (8-11)), we find:

$$\ln\langle P(r,N)\rangle_{N_{\rm av}} \sim \begin{cases} -(r/\xi_r)^{\delta_r}\,, & r_1^{\rm B} < r < r_\times^{\rm B}(N_{\rm av}), \\ -r_c^{\rm B}(N_{\rm av})^{\delta_\ell(1-d_{\min})}\,(r/\xi_r)^{\delta_\ell d_{\min}}, & r > r_\times^{\rm B}(N_{\rm av}), \end{cases} \tag{21}$$

with $\delta_r = 1/(1-\nu_r)$, $\xi_r^{d_{\min}} = \xi_\ell[C_2^B/(\delta_\ell(d_{\min}-1))]^{(1-\nu_\ell)}$, and the limits $r_1^{\rm B}$ and $r_\times^{\rm B}$ correspond to r_1 and $r_\times$ from (9) and (10). Equation (21) represents only the exponential part of $\langle P(r,N)\rangle_{N_{\rm av}}$ which describes the asymptotic regime $r > r_1^{\rm B}$. For $r < r_1^{\rm B}$, we expect that $\langle P(r,N)\rangle_{N_{\rm av}}$ displays power-law dependence in the scaling variable $r/\xi_r \sim r/N^{\nu_r}$. Hence, in order to describe both the short-distance and the intermediate-distance regime, we infer the more general form

$$\langle P(r,N)\rangle \sim r^{-1}(r/\xi_r)^{g_r}\ \exp[-(r/\xi_r)^{\delta_r}], \qquad r < r_\times^{\rm B}(N_{\rm av}). \tag{22}$$

Figure 6 shows $r\langle P(r,N)\rangle$ versus r/N^{ν_r}. The figure supports our scaling result (22) with $g_r = 2.9 \pm 0.2$. Since $\langle P(r=1,N)\rangle = \langle P(\ell=1,N)\rangle$, the exponents g_ℓ and g_r describing the small ℓ- and r-behavior in Eqs. (16b) and (22) can be related to each other by

$$g_r = g_\ell\ d_{\min}. \tag{23}$$

Our numerical results yield $g_\ell = 2.5 \pm 0.2$ and $g_r = 2.9 \pm 0.2$, in good agreement with this prediction.

The exponent $\delta_\ell d_{\min} \cong 8.7$ for $r > r_\times^{\rm B}(N_{\rm av})$ in (21) is much larger than the exponent $\delta_r \cong 4.3$ in (21) [27]. This enhanced decay, however, cannot be seen in the figure, since $r_\times^{\rm B}(N_{\rm av}) > r_\times(N_{\rm av}) \simeq 2\xi_r$, for $N_{\rm av} = 1$ and we can only reach r values below $r \simeq 2\xi_r$ with our data.

We wish to thank the Deutsche Forschungsgemeinschaft and the Alexander von Humboldt-Stiftung for financial support.

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Fig. 1: Plot of the histogram $H(-\ln p)$ versus $-\ln(p)$ for $r = 70$ and $t = 1000$ (broad curve) for two dimensional percolation clusters at criticality. The smooth line represents the theoretical prediction [15]. The narrow curve represents the histogram in ℓ-space, for $\ell = 150$ and $t = 1000$. The large fluctuations in r-space are in marked contrast to the very narrow fluctuations in ℓ space. For obtaining the histograms, $p_i(\ell, t)$ and $p_i(r, t)$ were enumerated on $5 \cdot 10^3$ percolation clusters.

Fig. 2: **(a)** The minimum distance $\ell_{\min}(r, N_{\rm av})$ versus r for: RW-structures on the s.c.-lattice, ($N_{\rm av} = 1$ (full circle), 5 (square), 50 (full triangle) and 10000 (triangle)) (upper set of points, shifted by a factor 100 for clarity); bond percolation on the simple cubic lattice ($N_{\rm av} = 1$ (full circle), 10 (square) and 1000 (full diamond)) (intermediate set of points, shifted by a factor 10 for clarity), and site percolation on the square lattice ($N_{\rm av} = 1$ (full circle), 100 (square) and 250000 (full triangle) (lower set of points). For $N_{\rm av} < 10^4$, averages have been performed over typically 100 sets of $N_{\rm av}$ configurations. **(b)** Scale plot of $\ell_{\min}/r_c$ versus r/r_c for the same structures and the same $N_{\rm av}$ values shown in (a). Here, the same shifts as in (a) have been used (after [25]).

Fig. 3: **(a)** Plot of $-\ln(\langle p(r,t)\rangle_{N_{\rm av}}/\langle p(0,t)\rangle_{N_{\rm av}})$ for random walks versus $r/\langle R(t)\rangle$ for site percolation clusters on the square lattice ($t = 400$, $N_{\rm av} = 5$ (square) and 50 (full triangle), compared to the typical probability density (full circle) corresponding to the case $N_{\rm av} = 1$ (see [25]); $\langle R(t)\rangle$ is the r.m.s. displacement, and the straight lines have the theoretical slopes $d_\psi = 1.53$, for $r < r_\times$ and $d_\psi = 1.86$, for $r > r_\times$ (after [25]). **(b)** Plot of $-\ln\langle|u(r,\omega)|/|u(0,\omega)|\rangle$ versus $r/\xi_\ell^{1/d_{\rm min}}$ for the "typical" average, corresponding to the case $N_{\rm av} = 1$ (see [25]), of a fracton ensemble of frequencies $\omega \cong 0.1 f/M$, generated on site percolation clusters at criticality on a square lattice. The different clusters contained between 15000 and 30000 particles. Details of the calculations as well as the determination of $\xi_\ell \sim 29$ can be found in [5]. The straight line is shown as a guide to the eye.

Fig. 4: **(a)** The normalized mean probability density $\langle p(r,t)\rangle/\langle p(0,t)\rangle$ of random walks versus $(r/\xi_\ell^{1/d_{\rm min}})^g$ for site percolation clusters on the square lattice ($g \cong 1.35$, t=10 000, $\xi_\ell \cong 40$, $N_{\rm av} = 100$). **(b)** The normalized mean amplitudes $\langle|\ u(r,\omega)\ |\rangle/\langle|$

$u(0,\omega)\ |\rangle$ for fractons versus $(r/\xi_\ell^{1/d_{\min}})^g$ for site percolation clusters on the square lattice ($g \cong 1.35$, $N_{\rm av} \cong 50$) for two frequencies: $\omega^2 \cong 0.005 f/M$ ($\xi_\ell \cong 21$) (circles) and $\omega^2 \cong 0.009 f/M$ ($\xi_\ell \cong 17$) (full triangles). The full lines in the plots represent the theoretical predictions, Eq. (15), without any fit parameters (after [26]).

Fig. 5: Plot of $\ell\langle P(\ell,N)\rangle$ versus ℓ/N^{ν_ℓ} on a double logarithmic graph, for SAWs on two-dimensional percolation clusters at criticality grown on a square lattice up to 400 chemical shells. The circles represent the numerical data for $N = 100$ and the triangles for $N = 200$. The dashed line represents the functional form of Eq. (16a) with $\nu_\ell = 0.87 \pm 0.02$, $\delta_\ell = 1/(1-\nu_\ell)$, $g'_\ell = 3.0 \pm 0.2$, and $\xi_\ell = 1.27\ N^{\nu_\ell}$. For $\ell/N^{\nu_\ell} < 0.35$, a smaller value of the exponent g_ℓ represents the data better and we find $g_\ell = 2.5 \pm 0.2$ (Eq. 16b) (after [27]).

Fig. 6: Plot of $r\langle P(r,N)\rangle$ versus r/N^{ν_r} for the SAWs considered in Fig. 5, on a double logarithmic graph. The squares represent the numerical data for $N = 100$ and the triangles for $N = 200$. The line represents the analytical function, Eq. (22), which fits the data for all r/N^{ν_r}, with $\nu_r = 0.76 \pm 0.02$, $\delta_r = 1/(1-\nu_r)$, $g_r = 2.9 \pm 0.2$, and $\xi_r = 1.04\ N^{\nu_r}$ (after [27]).