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Properties of Lévy flights on an interval with absorbing boundaries

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Abstract

We consider a Lévy flyer of order α that starts from a point x_0 on an interval $[0, L]$ with absorbing boundaries. We find a closed-form expression for an arbitrary average quantity, characterizing the trajectory of the flyer, such as mean first passage time, average total path length, probability to be absorbed by one of the boundaries. Using fractional differential equations with a Riesz kernel, we find exact analytical expressions for these quantities in the continuous limit. We find numerically the eigenfunctions and the eigenvalues of these equations. We study how the results of Monte-Carlo simulations of the Lévy flights with different flight length distributions converge to the continuous approximations. We show that if x_0 is placed in the vicinity of absorbing boundaries, the average total path length has a minimum near $\alpha = 1$, corresponding to the Cauchy distribution. We discuss the relevance of these results to the problem of biological foraging and transmission of light through cloudy atmosphere.

1. Introduction

In the past two decades, Lévy flights (see Refs. [1–11]) found numerous applications in natural sciences. Realizations of Lévy flights in physical phenomena are very diverse, including fluid dynamics, dynamical systems, and statistical mechanics. Recently, Lévy

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flights have been proposed as a model for animal foraging [7,12–16], cell diffusion [17], and transmission of light through cloudy atmosphere [18,19].

In general, Lévy flights model anomalous diffusion, which is governed by rare but extremely large jumps of diffusing particles. Lévy flights are characterized by broad distributions of their step lengths, for which the second moment does not exist. Lévy flights of order $\alpha < 2$ have distributions of step lengths with diverging moments of order $m \geqslant \alpha$ and converging moments of order $m < \alpha$. In the absence of absorbing boundaries, the generalized central limit theorem [4,20] guarantees that the probability density $\Psi(x,n)$ of the displacement x of Lévy flights converges after many steps to the Lévy stable distribution of order α:

$$\Psi(x,n) = \frac{1}{\pi}\int_0^\infty \exp(-n\ell_0^\alpha q^\alpha)\cos(qx)\,\mathrm{d}q\,, \tag{1}$$

where ℓ_0 is the characteristic width of the distribution of a single step and n is number of steps. This distribution is a generalization of the Gaussian distribution, and is characterized [4] for $x \to \infty$ by the power law decay of its density

$$\Psi(x,n) = n\ell_0^\alpha \gamma_1 x^{-\alpha-1} + O(x^{-2\alpha-1})\,, \tag{2}$$

where

$$\gamma_1 = \pi^{-1}\Gamma(\alpha+1)\sin(\pi\alpha/2)\,.$$

In the absence of boundaries, the generalized central limit theorem allows us to treat Lévy flight diffusion with the help of fractional differential equations [3,4,6,9,21–26]. In the presence of boundaries, the fractional derivative formalism is less clear. However, this case has many practical applications, in which it is important to find the mean first passage time, the average trajectory length and other quantities of interest. The behavior of these quantities is important in studies of biological foraging [15,16], in which the foraging efficiency is defined to be inversely proportional to total path length. Lévy flights in a slab geometry with absorbing boundaries have been used to model the transmission of light through cloudy atmosphere [18]. Using heuristic arguments confirmed by numerical simulations, Ref. [18] found the scaling behavior of the transmission probability of a photon through a slab of width L and the total geometrical path length of transmitted and reflected light. This behavior was experimentally observed in Ref. [19].

The average total path length traveled by a Lévy flyer before absorption, $\langle S(x_0)\rangle$, is equivalent to the total time spent by a Lévy walker before absorption. Lévy walks on a finite interval with absorbing boundaries has been studied in Ref. [27], where the asymptotic behavior of the survival probability was found. This behavior is related to the asymptotic behavior of $\langle S(x_0)\rangle$. Recently [28], the approximate expressions for the mean first passage time for both Schneider–Wyss sub-diffusion equation [29] and superdiffusion equation [22] have been obtained by separation of variables. The latter case exactly corresponds to the Lévy flight problem. Later [30], the exact expressions for the mean first passage time and other average quantities for Lévy flights have been derived with the help of fractional differential equations with Riesz kernels [3,21,31–33]. Here, we extend the results of Refs. [28,30] and test them by Monte-Carlo (MC) simulations.

2. Discrete Lévy flights

Consider a Lévy flight that starts at point x_0 of the interval $[0, L]$ with absorbing boundaries. The flyer makes independent subsequent flights of variable random lengths ℓ with equal probability in both directions. The length of each flight is taken from the power-law distribution

$$P(|\ell| > r) = (\ell_0/r)^\alpha , \tag{3}$$

where exponent α can vary between 0 and 2, and ℓ_0 is the minimal flight length, which serves as the lower cutoff of the distribution. The probability density of the flight length is given by

$$p(\ell) = \frac{\alpha\ell_0^\alpha}{2}\,\frac{\theta(|\ell| - \ell_0)}{|\ell|^{\alpha+1}} , \tag{4}$$

where $\alpha\ell_0^\alpha/2$ is a normalization constant and $\theta(x) = 1$ for $x > 0$ or 0 otherwise. When $\alpha > 2$, the second moment of the flight distribution converges and the process becomes equivalent to normal diffusion. As soon as the flyer lands outside the interval $[0, L]$, the process is terminated. Instead of the probability density Eq. (4), one could use any power-law decaying density [4,27] regularized at small distances ℓ_0, including Lévy stable distribution Eq. (1) with $n = 1$.

Suppose that the probability density of finding a Lévy flyer at point x after n flights is $P_n(x)$. Then the probability density after $n+1$ flights is given by the convolution of the probability density $P_n(x)$ and the probability density of the next flight $p(x)$ given by Eq. (4)

$$P_{n+1}(y) = \int_0^L p(y - x)P_n(x)\,\mathrm{d}x . \tag{5}$$

Let $\mathscr{L}_\alpha(\ell_0)$ be an integral operator with kernel $p(x - y)$ which is defined on a function $f(x)$ of an interval $[0, L]$. Then the distribution after n flights is given by

$$P_n(x) = [\mathscr{L}_\alpha^n P_0](x) . \tag{6}$$

The initial probability density of the flyer located at position x_0 is the Dirac delta function, $P_0(x) = \delta(x - x_0)$.

In general, consider a quantity $\langle Q(x_0)\rangle = \langle\sum_{i=1}^{\infty} q_i\rangle$, where $q_i = q(x_{i-1}, x_i)$ is a function of the starting point x_{i-1} and ending point x_i of the ith flight, and $\langle\ \rangle$ denotes the average over all possible processes starting at point x_0. Then, such a quantity [30] must satisfy a recursion relation

$$\langle Q(x_0)\rangle = \langle q_0(x_0)\rangle + \int_0^L \langle Q(x_1)\rangle\, p(x_0 - x_1)\,\mathrm{d}x_1 , \tag{7}$$

where $\langle q_0(x)\rangle \equiv \int_{-\infty}^{\infty} p(x_1 - x_0)q(x_0,x_1)\,dx_1$. Note that if x_1 is outside the interval $[0,L]$, the particle is absorbed by one of the boundaries and the value $q(x_0,x_1)$ should be defined according to its physical meaning for the absorbed particle. Eq. (7) belongs to the class of Fredholm integral equations of the second kind [34], and has a unique solution

$$\langle Q(x_0)\rangle = (\mathscr{I} - \mathscr{L})^{-1}\langle q_0(x_0)\rangle\,, \tag{8}$$

where $\mathscr{I}$ is a unity operator $\mathscr{I} f(x) \equiv f(x)$.

The most simple example of the quantity $\langle Q(x_0)\rangle$ is the average number of steps before absorption $\langle n(x_0)\rangle$. In this case $\langle q_0(x_0)\rangle = 1$. Another example is the total flux through the right boundary. This flux is related to the transmission probability of photons through the clouds [18]. By definition, the flux through the right boundary is equal to the probability $P_r(x_0)$ of the absorption of the flyer that starts at point x_0 by the absorbing boundary $x=L$. In this case, quantity q must be defined as $q(x_0,x_1)=\theta(x_1-L)$. The very first flight is absorbed by the right boundary with probability $p_r(x_0) \equiv \langle q_0(x_0)\rangle = \int_L^{\infty} p(x_1 - x_0)\,dx_1$, or after integration for the case of truncated power-law density and $x_0 \leqslant L - \ell_0$

$$p_r(x_0) = [\ell_0/(L - x_0)]^{\alpha}/2\,. \tag{9}$$

Therefore, $P_r(x_0)$ satisfies Eq. (7) with $\langle q_0(x_0)\rangle = p_r(x_0)$. For the case of Lévy stable density, one cannot write down such a simple expression for $p_r(x_0)$, but as it follows from Eq. (2), the asymptotic behavior of $p_r(x_0)$ for $\ell_0 \to 0$ remains the same as in Eq. (9) with the only difference that now it is multiplied by the coefficient $2\gamma_1/\alpha$, i.e., $p_r(x_0) = \gamma_1[\ell_0/(L - x_0)]^{\alpha}/\alpha + O([\ell_0/(L - x_0)]^{2\alpha})$.

To find the expression for the average total path length, one must define $\langle q_0(x_0)\rangle \equiv s(x_0)$, where $s(x_0)$ is the average length of the first flight that starts from a point x_0. In the absence of the absorbing boundaries, the average flight length with probability density $p(\ell)$ of Eq. (4) is given by $\langle|\ell|\rangle = \int_{-\infty}^{+\infty} |\ell|\,p(\ell)\,d\ell$, which is independent of the starting point and diverges for $\alpha \leqslant 1$. In the presence of the absorbing boundaries, the flight starting from a point x_0 cannot exceed the distances x_0 and $L - x_0$ from this point to the boundaries. One can show that the average length of a flight that starts from a point x_0 of an interval $[\ell_0, L - \ell_0]$ is given in case of truncated power-law density by

$$s(x_0) = \frac{\ell_0}{2(1-\alpha)}\left[\left(\frac{\ell_0}{x_0}\right)^{\alpha-1} + \left(\frac{\ell_0}{L - x_0}\right)^{\alpha-1} - 2\alpha\right]\,. \tag{10}$$

If $\alpha = 1$, this expression has to be replaced by the expression containing logarithms [30]. In case of the Lévy stable density we can find the asymptotic behavior of this expression in case $\alpha > 1$, i.e., $s(x_0) = 2\int_0^{\infty} \Psi(x,1)x\,dx = 2\Gamma(1 - 1/\alpha)/\pi$ [4]. In case $\alpha < 1$, the asymptotic behavior is given by the same expression Eq. (10), in which the first two power-law terms are multiplied by the constant $2\gamma_1/\alpha$, and the last constant term is neglected.

Finally, one can find the average number $\langle n_r(x_0)\rangle$ and the average total path length $\langle S_r(x_0)\rangle$ of the flights absorbed by the right boundary. One can show, that the quantity $P_r(x_0)\langle n_r(x_0)\rangle$ satisfies Eq. (7) with $\langle q_0(x_0)\rangle = \int_0^L P_r(x_1)p(x_1-x_0)\,\mathrm{d}x_1 + p_r(x)$, and the quantity $P_r(x_0)\langle S_r(x_0)\rangle$ satisfies Eq. (7) with $\langle q_0(x_0)\rangle = \int_0^L P_r(x_1)p(x_1-x_0)|x_1-x_0|\,\mathrm{d}x_1 + (L-x_0)p_r(x)$. These quantities are of interest for the problem of light propagation through the clouds [18,19].

3. The continuous limit

It can be shown that for the case of the power-law truncated kernel with $\alpha < 2$ [30], there exists an operator [3,31,32]

$$\mathscr{D}_\alpha \equiv \lim_{\ell_0\to 0} \ell_0^{-\alpha}[\mathscr{L}_\alpha(\ell_0) - \mathscr{I}]\,. \tag{11}$$

The result of this operator acting on any function $f(x)$ with a derivative $f'(x)$ and finite values at the boundaries $x=0$ and L is defined as

$$[\mathscr{D}_\alpha f](x) = \mathrm{V.P.}\int_0^L \frac{\mathrm{sgn}(y-x)f'(y)\,\mathrm{d}y}{2|y-x|^\alpha} - \frac{f(0)}{2x^\alpha} - \frac{f(L)}{2(L-x)^\alpha}\,. \tag{12}$$

This operator is a self-adjoint operator similar to the double differentiation operator $\mathrm{d}^2/\mathrm{d}x^2$. It can be expressed [3,31] as the linear combination of right and left Riemann–Liouville fractional derivatives of the order α. In case of power-law truncated kernel [30], the difference of the two operators

$$d_\alpha(\ell_0) \equiv \ell_0^{-\alpha}[\mathscr{L}_\alpha(\ell_0) - \mathscr{I}] - \mathscr{D}_\alpha \tag{13}$$

decays as $\ell_0^{2-\alpha}$ when $\ell_0 \to 0$. It can be shown that the leading term of the operator d_α is proportional to the operator of the second derivative

$$d_\alpha(\ell_0) = \ell_0^{2-\alpha}\frac{\alpha}{2(\alpha-2)}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + o(\ell_0^{2-\alpha})\,. \tag{14}$$

In the general case, when the kernel $p(|x-y|)$ of the operator $\mathscr{L}$ on the interval $[0,L]$ is such that $p(x)>0$, $\int_{-\infty}^{+\infty} p(|x-y|)\,\mathrm{d}x = 1$, and it has asymptotic behavior $p(x) = \gamma x^{\alpha+1}[1+o(1)]$ for $x\to\infty$, we still can expect that Eq. (11) is valid if the characteristic width $\ell_0 \equiv (2\gamma/\alpha)^{1/\alpha} \to 0$. However, the correction d_α may differ from Eq. (14), depending on the correction terms in the asymptotic for $p(x)$. This general theorem is of principal importance for the theory of Lévy flights on the bounded domain.

In analogy with the diffusion equation with continuous time, we can define a superdiffusion equation [3,21,22,28] based on Lévy flights. Instead of the discrete process defined by Eq. (5), one can write

$$\frac{\partial P(x,t)}{\partial t} = \frac{\ell_0^\alpha}{t_0}\mathscr{D}_\alpha P(x,t)\,, \tag{15}$$

where t_0 is the duration of each flight, and ℓ_0^α/t_0 is the fractional analog of the diffusion coefficient. Note that ℓ_0 plays a role similar to the mean free path, and t_0 plays the role of the mean collision interval.

The operator $\mathscr{D}_\alpha$ has an orthogonal normalized set of eigenfunctions $f_k(x)$, such that $\mathscr{D}_\alpha f_k(x) = \lambda_k f_k(x)$, and $f_k(0) = f_k(L) = 0$ [32]. Similarly to the solution of usual diffusion equation, the solution of Eq. (15) can be expressed via separation of variables as a series of eigenfunctions

$$P(x,t) = \sum_{k=1}^{\infty} e^{\lambda_k \ell_0^\alpha t/t_0} f_k(x) \int_0^L f_k(y) P(y,0)\, dy\,. \tag{16}$$

In Ref. [28], where the method of separation of variables for the superdiffusion equation on a finite interval has been first proposed, it has been assumed that the eigenvalues λ_k asymptotically behave at large k as $\lambda_k \sim -(k/L)^\alpha < 0$ and that the eigenfunctions $f_k(x)$ can be well approximated by the eigenfunctions $\sqrt{2/L}\sin(x\pi k/L)$ of the Laplace operator with absorbing boundary conditions. Numerical studies (see Section 5) confirm these assumptions but show that eigenfunctions f_k and sines have different behavior near absorbing boundaries, namely, $f_k(x) \sim x^{\alpha/2}$ as $x \to 0$.

4. Analytical expressions for average quantities

Having defined the properties of the operator $\mathscr{D}_\alpha$, we can derive continuous limit approximations $\bar{Q}(x_0)$ for the average quantities $\langle Q(x_0)\rangle$, defined for the discrete Lévy flights by Eq. (8). Formal substitution of Eq. (11) into Eq. (8) yields

$$\bar{Q}(x_0) = \mathscr{D}_\alpha^{-1} h(x_0)\,, \tag{17}$$

where function $h(x)$ satisfies the equation

$$h(x_0) = -\,\ell_0^{-\alpha}\langle q_0(x_0)\rangle\,. \tag{18}$$

In the general case, the equation

$$\text{V.P.} \int_0^L \frac{\operatorname{sgn}(y-x) f'(y)\, dy}{2|y-x|^\alpha} = h(x) \tag{19}$$

with boundary conditions

$$f(0) = a, \qquad f(L) = b \tag{20}$$

belongs to a known class of generalized Abel integral equations with Riesz fractional kernel [31–33]. It can be shown [32,33] that such an equation with boundary conditions (20) has a unique solution which can be obtained via spectral relationships for Jacobi polynomials [31,33] or by the Sonin inversion formula [33]. Similar inversion formulae are given in Ref. [32]. In particular [30,33], an equation:

$$\int_0^L \frac{\operatorname{sgn}(y-x)}{2|x-y|^\alpha} \varphi(y)\, dy = h(x), \quad 0 \leqslant x \leqslant L \tag{21}$$

has a general solution:

$$\varphi(x) = C\varphi_0(x) + \varphi_1(x)\,, \tag{22}$$

where $\varphi_0 = (Lx - x^2)^{(\alpha/2)-1}$ is a general solution of the homogeneous equation with $h(x) = 0$ and

$$\varphi_1(z) = \frac{4\sin(\pi\alpha/2)}{\pi\alpha \mathrm{B}(\alpha/2,\alpha/2)} z^{\alpha/2-1} \frac{\mathrm{d}}{\mathrm{d}z} \int_z^L \mathrm{d}t\, t^{1-\alpha}(t-z)^{\alpha/2} \frac{\mathrm{d}}{\mathrm{d}t} \times \int_0^t y^{\alpha/2}(t-y)^{\alpha/2-1} h(y)\,\mathrm{d}y \tag{23}$$

is a partial solution of the inhomogeneous equation. Since Eq. (19) contains $f'(x) = \varphi(x)$, one can always satisfy the first boundary condition Eq. (20) by defining $f(x) = \int_0^x \varphi(z)\,\mathrm{d}z + a$. The second boundary condition Eq. (20) can be satisfied if we select constant $C = L^{1-\alpha}\left[b - a - \int_0^L \varphi(x)\,\mathrm{d}x\right] / \mathrm{B}(\alpha/2,\alpha/2)$.

The probability of absorption by the right boundary $P_r(x_0)$ satisfies boundary conditions $a=0$, $b=1$. All other functions of interest satisfy absorbing boundary conditions $a=b=0$.

Using the Sonin inversion formula, one can obtain [30] the analytical continuous limit approximations $\bar{n}$, $\bar{S}$, and $\bar{P}_r$ for the average quantities $\langle n\rangle$, $\langle S\rangle$, and P_r, respectively. In the case of average number of steps, we have $h(x) = -\ell_0^{-\alpha}$ and the solution can be expressed in elementary functions:

$$\bar{n}(x_0) = \frac{\sin(\pi\alpha/2)}{\pi\alpha/2}\left[\frac{(L-x_0)x_0}{\ell_0^2}\right]^{\alpha/2} = \frac{\sin(\pi\alpha/2)M^\alpha(z-z^2)^{\alpha/2}}{\pi\alpha/2}\,, \tag{24}$$

where $z \equiv x_0/L$, $M \equiv L/\ell_0$. One can verify this solution by performing contour integration around the cut $[0,L]$ on the complex plane and computing the residue of the integrand at infinity. Note that in case of Lévy stable density, this equation has to be divided by the normalization coefficient $2\gamma_1/\alpha$, and thus we have $\bar{n}(x_0) = M^\alpha(z - z^2)^{\alpha/2}/\Gamma(\alpha+1)$.

In case of the average total path length, Sonin formula leads to an expression

$$\bar{S}(x_0) = L[A(\alpha,z) + B(\alpha,z)M^{\alpha-1}]\,, \tag{25}$$

where the first term dominates for $\alpha < 1$ and the second term dominates for $\alpha > 1$. In case of truncated power-law density, this terms can be found explicitly:

$$A(\alpha,z) = \frac{(2-\alpha)}{2(1-\alpha)}\left[1 - 4\frac{\psi_\alpha(z)+\psi_\alpha(1-z)}{\alpha(\alpha+2)\mathrm{B}(\alpha/2,\alpha/2)}\right]\,, \tag{26}$$

where $\psi_\alpha(z) = F(2-\alpha/2, \alpha/2, \alpha/2+2, z)z^{\alpha/2+1}$ is a hypergeometric function [35,36] and

$$B(\alpha,z) = \frac{2\sin(\pi\alpha/2)(z-z^2)^{\alpha/2}}{\pi(\alpha-1)}\,. \tag{27}$$

In case of Lévy stable density the asymptotic for $\alpha < 1$ remains the same, but the asymptotic for $\alpha > 1$ changes: $B(\alpha, z) = 2\pi^{-1}(z - z^2)^{\alpha/2}\Gamma(1 - 1/\alpha)/\Gamma(1 + \alpha)$.

Finally, the probability $\bar{P}_r(x_0)$ of the absorption by the right boundary in the continuous limit is given by the solution of the homogeneous equation (21)

$$\bar{P}_r(x_0) = \left(\frac{x_0}{L}\right)^{\alpha/2} \frac{F(\alpha/2, 1 - (\alpha/2), (\alpha/2) + 1, x_0/L)}{(\alpha/2)\mathrm{B}(\alpha/2, \alpha/2)} . \tag{28}$$

For $x_0 \to 0$, the asymptotic behavior of $\bar{P}_r(x_0)$ is given by $\bar{P}_r(x_0) \sim (x_0/L)^{\alpha/2}$ which is in complete agreement with the result of Ref. [18] for the transmission probability of the photons through the clouds of depth L. For Lévy stable density, the asymptotic expression for $\bar{P}_r(x_0)$ remains the same.

5. Numerical analysis

In order to obtain the eigenvalues and the eigenfunctions of the operator $\mathscr{D}_\alpha$, we extrapolate to $M \to \infty$ the eigenvalues and the eigenvectors of the $M \times M$ symmetrical matrix $\mathscr{A}_{ij} = p(|i - j|)$, where $p(j)$ are the transition probabilities for the Riemann random walk [1]. This matrix is the discrete analog of the operator $\mathscr{L}$. Riemann random walk performs flights of non-zero integer length $\pm j$ with probability $p(\pm j) = [2\zeta(\alpha + 1)j^{\alpha+1}]^{-1}$, where $\zeta(\alpha) \equiv \sum_{j=1}^{\infty} j^{-\alpha}$ is the Riemann ζ-function. The Riemann transition probabilities are slightly different from the ones treated in Ref. [30]. We choose the Riemann probabilities here, because they lead to faster convergence of the eigenvectors and the eigenvalues to the continuous limit. We test our result with transition probabilities of Ref. [30] and find good agreement for large values of M.

We find the first ten orthonormal eigenvectors $\mathbf{f}_k$ such that $\mathscr{A}\mathbf{f}_k = \mu_k \mathbf{f}_k$ for several increasing values of M. According to a general theorem which we mention above, the matrix $\ell_0^{-\alpha}(\mathscr{A} - \mathscr{I})$ should converge to $\mathscr{D}_\alpha$ on the interval $(0, 1)$ if $\ell_0 \equiv (\alpha\zeta(\alpha + 1))^{-1/\alpha}/M \to 0$. Indeed we find that $\mathbf{f}_k$ converge to the continuous eigenfunctions $f_k(x)$ defined on the interval $(0, 1)$ with $x = j/(M + 1)$, and $(\mu_k - 1)/[\alpha\zeta(\alpha + 1)M^\alpha] \to \lambda_k$, where λ_k are the eigenvalues of the operator $\mathscr{D}_\alpha$. Fig. 1 shows that the eigenfunctions of the operator $\mathscr{D}_\alpha$ resemble $\sqrt{2}\sin(k\pi x)$, the eigenfunctions of the classical diffusion equation. Moreover, for large values of the wave number k, the eigenfunctions f_k converge to sines except in the vicinity of the boundaries. This behavior is expected since in the center of the interval, the effect of the boundaries can be neglected and the problem becomes equivalent to the problem of the fractional diffusion on the infinite interval. The latter case can be solved by means of Fourier transform [22] and it was shown that $\sin(qx)$ are the continuous spectrum eigenfunctions with eigenvalues $-q^\alpha\Gamma(1 - \alpha)\cos(\pi\alpha/2)$ [31,32]. Following the logic of Ref. [28], one can expect that in the case of finite interval, we should have discrete spectrum with $q = k\pi/L$ in order to satisfy absorbing boundary conditions. Thus for large k, the eigenvalues λ_k should behave as

$$\lambda_k = -(k\pi/L)^\alpha\Gamma(1 - \alpha)\cos(\pi\alpha/2)[1 + O(1/k)] . \tag{29}$$

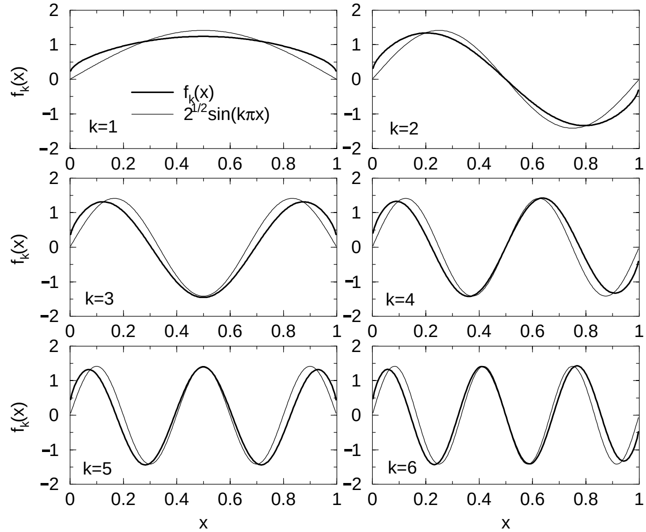

Fig. 1. First six eigenfunctions $f_k(x)$ for $\alpha = 0.5$ for $M = 1000$, $L = 1$ (bold lines) in comparison with $\sqrt{2}\sin(k\pi x)$ (thin lines).

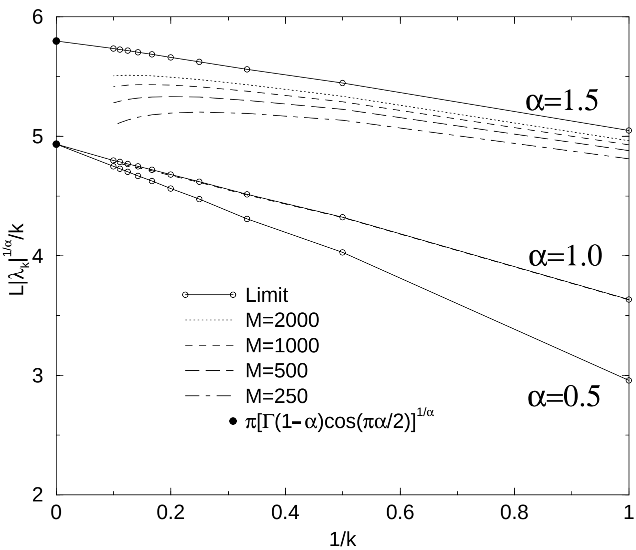

Fig. 2. The behavior of the scaled eigenvalues $L|\lambda_k|^{1/\alpha}/k$ versus $1/k$ for $\alpha = 0.5, 1.0, 1.5$ and increasing values of M. Circles on the x-axis indicate the predictions of Eq. (29).

This is illustrated by Fig. 2 which shows the quantities $L|\lambda_k|^{1/\alpha}/k$ versus $1/k$. Indeed, the graphs are well approximated by straight lines with intercepts that are close to the values predicted by Eq. (29). Note that for large α, convergence of the operator $\mathscr{L}$ becomes very slow. According to Eq. (14), we can expect that the eigenvalues for

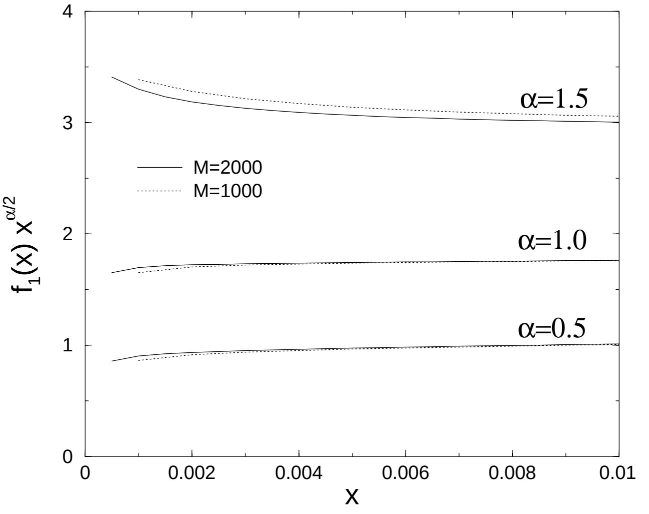

Fig. 3. The scaled behavior of the first eigenfunction $f_1(x)/x^{\alpha/2}$ versus x, for $\alpha=0.5, 1.0, 1.5$. for $M=1000$ (dotted line) and 2000 (solid line).

finite M approach limiting values as $M^{\alpha-2}$. For the case $\alpha=1.5$, shown on Fig. 2, the error decays as $M^{-0.5}$ and we can extrapolate the eigenvalues for $M\to\infty$. This limiting set of eigenvalues perfectly converges for $1/k\to 0$ to the value predicted by Eq. (29).

In order to demonstrate that in the vicinity of the absorbing boundaries, the eigenfunctions behave as $x^{\alpha/2}$, we plot the main eigenfunction $f_1(x)x^{-\alpha/2}$ for small values of x (see Fig. 3). One can see that the graphs are almost straight lines with small slopes and finite intercepts. Moreover, the quality of the straight line fits improves with increasing M. The rigorous analytical derivation of the properties of the eigenvalues and the eigenfunctions of the operator $\mathscr{D}_\alpha$ remains to be an open question.

Numerical tests of convergence of the discrete Lévy flights to the solution of the superdiffusion equation has been performed in Ref. [30] using matrix approach similar to the method for finding eigenfunctions described above. Here, we demonstrate the convergence using direct MC simulations of the Lévy flights with truncated power-law density Eq. (4) and Lévy stable density Eq. (1).

Fig. 4a illustrates, for the power-law density, the convergence of $\langle n\rangle$, $\langle S\rangle$, and P_r to the continuous analytical expressions $\bar{n}$, $\bar{S}$ and $\bar{P}_r$, respectively. One can see excellent convergence for $M\to\infty$ of the simulation results (symbols) to the analytical expressions (lines) for $\alpha<1$, but slow convergence for $\alpha\to 2$, where corrections decay as $M^{\alpha-2}$. Fig. 4b shows the behavior of the same quantities for the Lévy stable density simulated with help of Eq. (8.2.1) given in Ref. [4]. On can see excellent convergence for $\alpha>1$, but poor convergence for $\alpha\to 0$, when we can expect correction terms decaying as $M^{-\alpha}$.

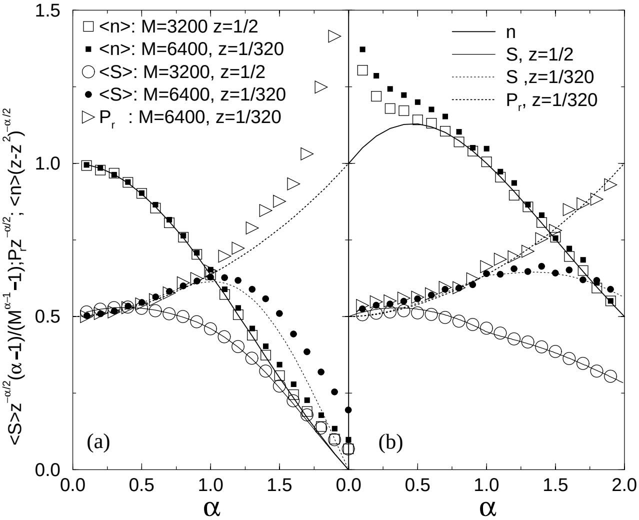

Fig. 4. The behavior of three scaled quantities $\langle n \rangle$, P_r, and $\langle S \rangle$ (symbols) obtained for $L=1$ and 10^5 MC realizations in comparison with scaled analytical solutions (lines) of Eqs. (24), (28), and (25), respectively. (a) Power-law truncated density, (b) Lévy stable density.

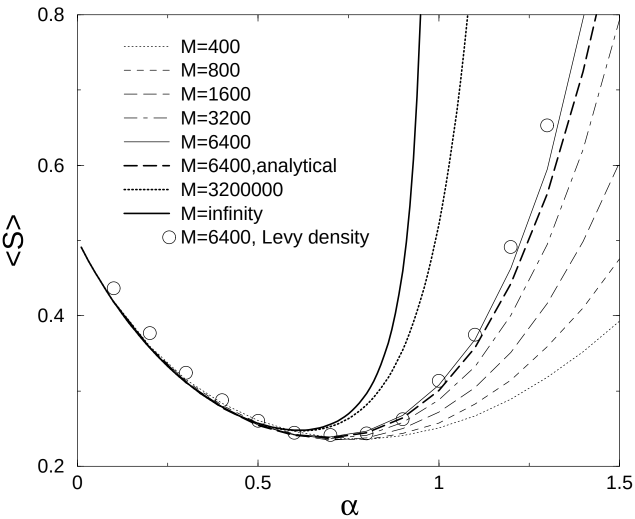

Fig. 5. The behavior of $\langle S \rangle$ versus α near its minimum for $L=1$, $z=1/320$ and various values of M, shown on the graph. Bold lines indicate analytical solutions. Thin lines indicate MC simulations with truncated power-law density. Circles indicate MC simulations with Lévy stable density.

Finally we test the convergence of the average quantities for $M \to \infty$ when x_0 is in the vicinity of absorbing boundary. The condition of this convergence is $x_0 \gg \ell_0$, or $zM \gg 1$. Fig. 5 shows the convergence of $\langle S \rangle$ for the case $z=1/320$. Already, for $M = 6400$ the convergence is good in a broad region of $0 < \alpha < 1.5$. For any finite z and $M \to \infty$, the values of $\langle S \rangle$ converge to $LA(\alpha, z)$. For $z \ll 1$, the expression $A(\alpha, z)$ asymptotically behave [30] as $2z^{\alpha/2}/[\alpha \mathrm{B}(\alpha/2, \alpha/2)(1-\alpha)]$. This asymptotic has a singularity at $\alpha = 1$ and a minimum at $\alpha = \alpha_{\min} = 1 + 2/\ln z + o(1/\ln z)$. For finite M, there is no singularity at $\alpha = 1$ and the minimum shifts to the left, closer to $\alpha = 1$. Thus, for large enough M, the minimum of the average total path length for discrete Lévy flights satisfies inequality $1 + 2/\ln z < \alpha_{\min} < 1$. Note that this behavior does not depend on the particular type of the distribution of the flights since the expression for $A(\alpha, z)$ is the same for all distributions decaying for large x as $x^{-\alpha-1}$. To illustrate this we include in Fig. 5 the graph for the Lévy stable density.

6. Summary

We have studied Lévy Flights in a finite interval with absorbing boundaries. We show that many quantities of interest obey general recursion relation Eq. (7) which has a form of the Fredholm integral equation of the second kind. We apply this method to derive the probability of absorption by one of the boundaries, the average number of flights and the average total path length of the Lévy flights which is equivalent to the mean first passage time of the Lévy walks.

We have shown how the discrete Lévy flights are related to the superdiffusion fractional differential equation (15). This equations can be solved by separation of variables, expressing solutions in series of eigenfunctions of Riesz operator Eq. (12). We have computed the eigenfunctions and eigenvectors of this operator numerically, and study their asymptotic properties.

For the continuous process described by Eq. (15), we derived exact analytical expressions Eqs. (24), (25), and (28) for the mean first passage time, the average total path length, and the probability of absorption by one of the boundaries, respectively. All these quantities are the solutions of the fractional differential equation (19) with Riesz kernel and with different right-hand sides. We have compared these analytical solutions with numerical solutions obtained for the discrete Lévy flights distributed with truncated power-law density and with Lévy stable density. We have shown that fractional differential formalism provides good approximation for the truncated power-law Lévy flights in the interval with absorbing boundaries except in the case of $\alpha \to 2$. In contrast, for Lévy stable density, the convergence is good for $\alpha \to 2$, but vanishes for $\alpha \to 0$.

We have shown that expression for the average total path length has a minimum at $\alpha \approx 1$ if the process starts in the vicinity of the absorbing boundaries (see Fig. 5). This result, as well as Eqs. (24), (25), and (28) can be applied to the problem of light transmission through cloudy atmosphere [17,18] and biological Lévy flight foraging in sparsely distributed food environment [15].

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