Self-avoiding walks on finitely ramified fractals

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We present an exact real-space renormalization-group technique for self-avoiding walks performed on finitely ramified fractals. The exponent $1/\nu$ of the walks is found to be strongly affected by the geometry of the particular fractal. Also, the excluded volume is shown to be relevant for some fractals and irrelevant for others. For the Sierpinski gasket $1/\nu$ is shown to be equal to the fractal dimensionality of the gasket.

Recently there has been an increasing interest in the problem of self-avoiding walks (SAW's) on fractal lattices and on percolation clusters.[1-4] These two problems are related since exact fractals are a good model for the backbone of the incipient infinite cluster at percolation.[5,6]

In this paper we present a real-space renormalization-group (RSRG) technique which is exact for SAW's on finitely ramified fractals. Finitely ramified fractals are fractals in which one can isolate a part of it of any linear size R by "cutting" at a minimal number $\mathcal{N}(R)$ of places, and $\mathcal{N}(R)$ remains bounded as $R\to\infty$. As a matter of fact, we study in here *exact* finitely ramified fractals where $\mathcal{N}(R)$ remains constant as R is increased by the rescaling factor of the fractal. This is what enables the RSRG to be analyzed exactly in this kind of problem for the first time.

We begin by presenting the RSRG technique on the Sierpinski gasket [Fig. 1(a)]. The technique is essentially a decimation procedure on the points 1, 2, and 3 of Fig. 1(a). Let P_1' be the total number of SAW's which can be performed between vertices A and B without reaching C, and P_2' the total number of SAW's which can be performed from A via C to B. These quantities can be exactly calculated from the knowledge of P_1 and P_2 which are similarly defined on a fractal smaller by the scaling factor 2 (e.g., on $A13$). In Figs. 1(b) and 1(c) we show the different configurations contributing to P_1' and P_2', thus

$$P_1'=P_1^2+P_1^3+2P_1P_2+P_2^2+2P_1^2P_2\ ,$$
$$P_2'=P_1^2P_2+2P_1P_2^2\ . \tag{1}$$

From Eqs. (1) one can iterate the ratio $\mu=P_1/P_2$ which converges extremely rapidly to $\mu\simeq 2.7036$. Let N_1 be the mean number of steps for a SAW to reach point 1 starting from A without passing through 3. Similarly, N_2 is defined for SAW's which pass through 3. Then it can be easily shown that in order to calculate the respective N_1' and N_2' (on the ABC fractal) one just has to weight the contribution of N_1 by P_1 and of N_2 by P_2 for each configuration. Thus

$$P_1'N_1'=(2P_1^2+2P_1P_2+3P_1^3+4P_1^2P_2)N_1+(2P_1P_2+2P_2^2+2P_1^2P_2)N_2\ , \tag{2}$$
$$P_2'N_2'=(2P_1^2P_2+2P_1P_2^2)N_1+(P_1^2P_2+4P_1P_2^2)N_2\ .$$

Note that the recursion formulas (1) and (2) are exact due to the finite ramification of the Sierpinski gasket, while in contrast for homogeneous space the best one can do is a finite cluster approximation.[7] Now using the fact that P_1 diverges while the ratio $P_1/P_2=\mu$ rapidly approaches a finite limit, one gets from a repeated iteration of Eqs. (1) and (2)

$$\begin{pmatrix}N_1'\\N_2'\end{pmatrix}=\begin{pmatrix}\dfrac{4+3\mu}{2+\mu} & \dfrac{2}{2+\mu}\\[2mm] \dfrac{2+2\mu}{2+\mu} & \dfrac{4+\mu}{2+\mu}\end{pmatrix}\begin{pmatrix}N_1\\N_2\end{pmatrix}\ . \tag{3}$$

The transforming matrix has the eigenvalues $\lambda_1=3$ and $\lambda_2=1$, thus for a SAW on the Sierpinski gasket

$$1/\nu=\ln\lambda_1/\ln b=\ln 3/\ln 2\simeq 1.585\ ,$$

where ν is the end-to-end exponent defined by $R(N)\approx\langle N\rangle^{\nu}$ and b is the rescaling factor. Note that $1/\nu$ is exactly equal to the fractal dimensionality of the gasket. This simple result can be understood on physical grounds. The peculiar topology of the gasket does not allow the excluded volume interaction to take effect since only one part of a SAW can enter and exit each subsection of the fractal due to the fact that there are only three ramification points. Thus only entropy considerations must be taken into account, and the maximal entropy is achieved for SAW's visiting all the parts of the fractal as seen from P_1' and P_2'.[8]

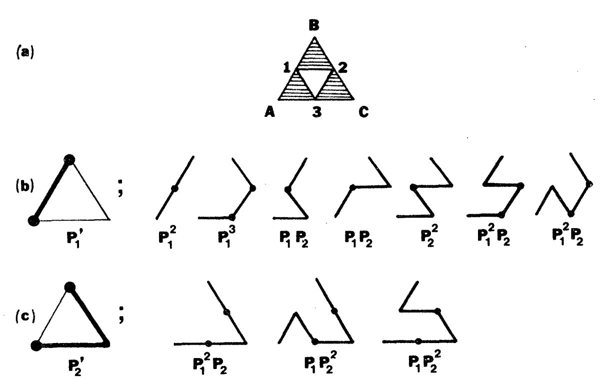

FIG. 1. (a) Sierpinski gasket. (b) and (c) Coarse graining of SAW's on the gasket.

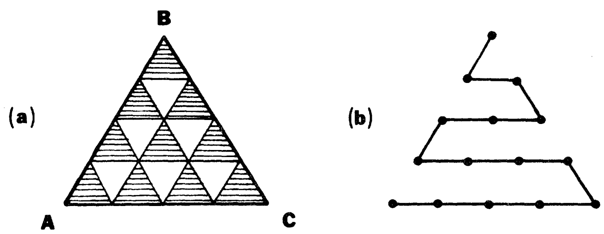

FIG. 2. (a) Generalization of the Sierpinski gasket (with $n=4$). (b) Possible SAW with $1/\nu=\bar{d}$.

We show that for the following generalization of the Sierpinski gasket similar results are derived. The generalization is simply based on an $n \times n$ checkboardlike triangle with n triangles belonging to the fractal on the base. The Sierpinski gasket fits the case of $n=2$. In Fig. 2(a) we show the case of $n=4$ as a further example. The fractal dimensionality is simply

$$\bar{d}_n = \frac{\ln[n(n+1)/2]}{\ln n} \ . \tag{4}$$

From the physical arguments discussed above and from the fact that one can always find paths [such as in Fig. 2(b)] which go through all the parts of these fractals it follows that for SAW's performed on them one gets $1/\nu=\bar{d}_n$.

However, the relation $1/\nu=\bar{d}_n$ is not true for all fractals, as can be seen from the Koch curvelike fractal presented in Fig. 3. Let P_1' be the number of all possible SAW's connecting A and B and passing through 1, and P_2' the number of those that do not pass through 1, then clearly

$$\begin{aligned} P_1' &= (P_1+P_2)^4 \ , \\ P_2' &= (P_1+P_2)^3 \ . \end{aligned} \tag{5}$$

Inserting the initial values $P_1=P_2=1$ and iterating Eq. (5) one sees that the configuration passing through point 1 (described by P_1) are dominant. Thus most of the SAW's pass through 4 out of the 5 parts of the fractal yielding $1/\nu=\ln 4/\ln 3 < d = \ln 5/\ln 3$.

Next we consider the fractal based on the iteration of Fig. 4(a). The three different cases of coarse graining are evident from Figs. 4(b), 4(c), and 4(d). The case in Fig. 4(d) is of special interest since only one of the branches (AB or CD) can be considered as belonging to the SAW while the other might be the consequence of an excluded volume effect. The quantity P_3' is defined as the number of all possible configurations of SAW's, one connecting the vertices A and B and the second connecting the vertices C and D so that they do not cross each other. The recursion relations are

$$P_1' = P_1 P_2^2, \quad P_2' = P_2^3, \quad P_3' = P_2^4 P_3 \ , \tag{6}$$

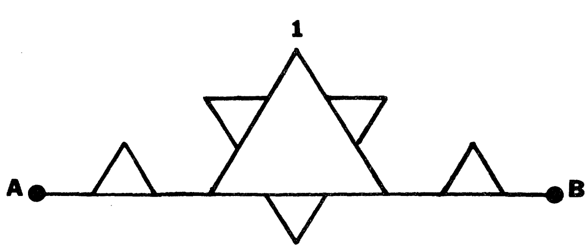

FIG. 3. Koch curvelike fractal with $\bar{d}=\ln 5/\ln 3$.

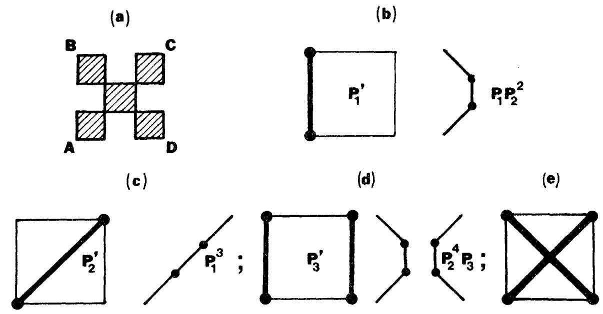

FIG. 4. (a) Checkboard fractal with $n=1$. (b), (c), and (d) Coarse graining of SAW's on the fractal. (e) Prohibited configuration because of excluded volume.

and, choosing the initial values $P_1=P_2=P_3=1$, one can see that these are also the final values after an arbitrary number of iterations. The obvious result is that for each possible configuration of a SAW, the SAW passes through just three of the subsections of the fractal, and since the rescaling factor is $b=3$ it means that the SAW is linear with $\nu=1$. Note that this result is not because of the excluded volume effect. One can clearly see that the contributions of the configurations in Figs. 4(b) and 4(d) are just the same in spite of the fact that in Fig. 4(d) there apparently is an excluded volume effect. However, the excluded volume makes such a configuration as in Fig. 4(e) impossible.

Finally, we consider the generalization of the fractal in Fig. 4(a) which is based on a checkboard of $(2n+1) \times (2n+1)$. Figure 4(a) is an example for $n=1$. We now discuss the case of $n=2$ [Fig. 5(a)]. The coarse graining can only yield typically the same three configurations as in the case of $n=1$. In Figs. 5(b), 5(c), and 5(d) we show some of the graphs which make the largest contributions to the corresponding P_s' defined as before. Thus

$$P_1' \sim 2P_1^5 P_2^4 P_3^2, \quad P_2' \sim 2P_1^4 P_2^5 P_3^2, \quad P_3' \sim 4P_1^4 P_2^4 P_3^5 \ . \tag{7}$$

It is clearly seen that P_3/P_1 and P_3/P_2 diverge upon iterat-

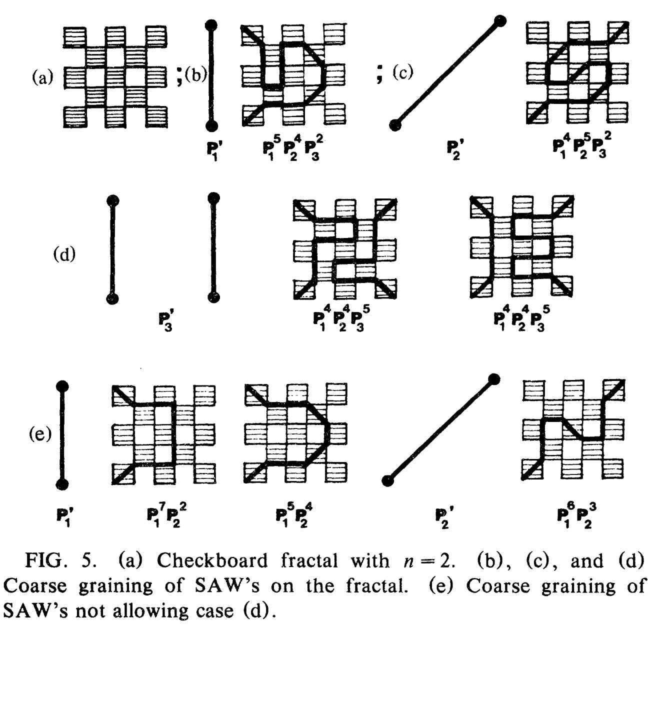

FIG. 5. (a) Checkboard fractal with $n=2$. (b), (c), and (d) Coarse graining of SAW's on the fractal. (e) Coarse graining of SAW's not allowing case (d).

ing, hence the configurations described by P_3 are dominant. Then one has to trace only N_3. But from Fig. 5(d) and the discussion above we get $N_3' = 5N_3$, or $\nu = 1$. Unlike the case of $n = 1$ this time it is the excluded volume effect that causes a decrease in $1/\nu$. This can be seen by setting $P_3 = 0$ (i.e., neglecting the excluded volume). Then the major contributions to the coarse graining of the cases in Figs. 5(b) and 5(c) are those given in Fig. 5(e). One easily gets then $1/\nu = \ln 9/\ln 5 > 1$. For a fractal checkboard with an arbitrary $n > 1$ it seems that P_3 will dominate the statistics, and then using configurations such as in Fig. 5(d) (which include as many cases of N_3 as possible) it turns out that

$$1 < 1/\nu_n = \frac{\ln(2n^2 - 2n + 1)}{\ln(2n+1)} < \bar{d}_n = \frac{\ln(2n^2 + 2n + 1)}{\ln(2n+1)} \; .$$

Thus the excluded volume in fractal checkboards can cause $1/\nu$ to drop below the dimensionality of the fractal.

In conclusion, we have presented a RSRG technique which seems to predict the exact value of the exponent ν of SAW's performed on finitely ramified fractals. The technique presented above enables the calculation of $1/\nu$ for SAW's on any finitely ramified fractal. We showed that a variety of cases may occur; the exponent $1/\nu$ ranges from 1 to the dimensionality of the embedding fractal. The excluded volume does affect SAW's on some fractals and does not affect SAW's performed on others. We used it to calculate $1/\nu$ for SAW's on the three-dimensional Sierpinski gasket and found its value to be 2 just as the fractal dimensionality of the gasket. We draw attention to the fact that the backbone of the infinite incipient cluster at the percolation threshold is believed to be modeled by a finitely ramified fractal.[5] Since the geometry of the fractal determines the value of the SAW exponent ν as shown in this work, the knowledge of ν (of the SAW) in percolation might provide valuable information about the geometrical structure of the infinite cluster.

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[1]R. Rammal and G. Toulouse, J. Phys. (Paris) Lett. 44, L213 (1983).

[2]B. K. Chakrabarti and J. Kertesz, Z. Phys. B 44, 221 (1981).

[3]K. Kremer, Z. Phys. B 45, 148 (1981).

[4]B. Derrida, J. Phys. A 15, 2119 (1982).

[5]Y. Gefen, A. Aharony, B. B. Mandelbrot, and S. Kirkpatrick, Phys. Rev. Lett. 47, 1771 (1981).

[6]S. Kirkpatrick, in *Ill-Condensed Matter,* Proceedings of the Les Houches Summer School, Session 31, edited by R. Bailian, R. Maynard, and G. Toulouse (North-Holland, Amsterdam, 1979).

[7]B. Shapiro, J. Phys. C 11, 2829 (1978).

[8]Since P_2 diverges and $\mu = P_1/P_2$ converges upon rescaling, it is obvious that only configurations involving P_1^3, $P_1^2 P_2$, $P_1 P_2^2$, or P_2^3 will contribute to the finite result. These configurations are the ones for which a SAW visits all parts of the fractal.