# Surface Roughening with Quenched Disorder in $d$-Dimensions

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We review recent numerical simulations of several models of interface growth in $d$-dimensional media with quenched disorder. These models belong to the universality class of anisotropic diode-resistor percolation networks. The values of the roughness exponent $\alpha = 0.63 \pm 0.01$ ($d = 1+1$) and $\alpha = 0.48 \pm 0.02$ ($d = 2+1$) are in good agreement with our recent experiments. We study also the diode-resistor percolation on a Cayley tree. We find that $P_\infty \sim \exp(-A/\sqrt{p_c - p})$, thus suggesting that the critical exponent for $P_\infty \sim (p_c - p)^{\beta_p}$, $\beta_p = \infty$ and that the upper critical dimension in this problem is $d = d_c = \infty$. Other critical exponents on the Cayley tree are: $\tau = 3$, $\nu_\| = \nu_\perp = \gamma = \sigma = 0$. The roughness exponents on the Cayley tree are: $\alpha = \beta = 0$, $z = 2$.

## 1. Introduction

The growth of rough interfaces in random media is a topic of current interdisciplinary interest [1-5]. For the most part, two types of models have been applied to interface-roughening phenomena:

(a) nonlinear Langevin-type equations—such as the KPZ equation [6]—resulting in *self-affine* interfaces, and

(b) spreading and invasion percolation models [7], generating *self-similar* interfaces

The self-affine interface (a) can be characterized by the rms surface width

$$w(\ell, t) \equiv \langle [h(x,t) - \langle h(x,t)\rangle]^2 \rangle^{1/2}. \tag{1}$$

Here $h(x,t)$ is the surface height at time $t$, and the angular brackets denote the average over $x$ belonging to a $d-1-$dimensional hypercube of size $\ell^{d-1}$ in the horizontal cross section perpendicular to the direction of growth.

An alternative and equivalent quantity is the height-height correlation function $c(\ell, t)$ [2]. The scaling exponents obtained from $w$ and $c$ are believed to be identical, so we use them interchangeably.

Analysis of the KPZ equation implies the scaling law [1-4]

$$w(\ell, t) \sim c(\ell, t) \sim \ell^\alpha f\left(\frac{t}{\ell^z}\right), \tag{2a}$$

where

$$f(u) \sim \begin{cases} u^\beta & u \ll 1 \\ \text{const} & u \gg 1, \end{cases} \tag{2b}$$

and

$$z = \alpha/\beta. \tag{2c}$$

Of course, the scaling function $f(u)$ is different for $w$ and for $c$. The roughness exponent is $\alpha = 1/2$ and the dynamical exponent is $\beta = 1/3$ for $d = 1 + 1$ [6]. Numerical studies of the KPZ equation give $\alpha \approx 2/(d+2)$ in $d \geq 2$ [8-10].

On the other hand, approach (b) (percolation-type models) produces *self-similar* interfaces [7] for which $w(\ell, \infty) \sim \ell^\alpha$ with $\alpha = 1$ and $\beta = 1$. For many phenomena in $d = 1 + 1$—from bacterial growth [11] and viscous flows [12,13] to the wetting [13–15], and burning [16] of paper—*self-affine* surfaces are found with anomalous exponents $\alpha$ and $\beta$ significantly larger than the KPZ values but less than 1. Recent experimental data in $d = 2 + 1$ also show anomalously large values of $\alpha$, e.g., for mountain surfaces $\alpha = 0.58$ [17,18], for wetting of porous media $\alpha = 0.5$ [19], and for ion beam erosion of metal surfaces $\alpha = 0.53$ [20].

An essential feature of many of these systems is that the noise is quenched, i.e. noise is a function of the interface position, as opposed to the uncorrelated noise of the KPZ equation. A critical question, then, is the scaling properties of systems with quenched noise. In particular, what phases, if any, are there which exhibit self-affine growth with $\alpha > \alpha_{\rm KPZ}$.

A complete answer to this question is not yet available. Recently, we and others have introduced an interesting class of models which exhibit new scaling behavior related to directed percolation. Another line of investigation [21–23] appears to indicate a direct transition in some models from a random-field Ising model phase with $\alpha = (5 - d)/3$ to KPZ behavior. However, more recent work [24] suggests that this result might be due to the neglect of a KPZ-type nonlinearity in the dynamics. As we shall see it is possible that the non-linear equation of Csahók et al. [24] is in the same universality class as our models. Clarifying the relationship of these and other models [25,26] is a topic that clearly merits more study.

As a step in answering this question, we develop and generalize several models [13,14,27–31] of spreading percolation with anisotropy in the growth direction which belong to a new universality class characterized by a self-affine interface with the roughness exponents given in Table 1. We explore the relation of this universality class to the reversed percolation transition point in the diode-resistor percolation network [28–30], at which the backflow current emerges in the direction opposite to the direction of diodes, when the concentration of resistors $q$ approaches some critical value $q_c = 1 - p_c$ ($p = 1 - q$ is the concentration of diodes). In $d = 1+1$, this transition point is dual to the directed percolation point [28–30] and the roughness exponent $\alpha$ of the interface between blocked and unblocked diodes can be expressed in terms of correlation exponents of directed percolation.

In $d > 2$, little is known about the critical properties of this transition. Recently, several theoretical conjectures have appeared concerning the values of the critical exponents of the surface roughening models with depinning transition caused by the quenched noise for continuum systems [24,25]. The numerical data for the continuum models in $d = 1 + 1$ strongly support the argument that these models belong to the same universality class as discrete models described below. However, according to [24,25] $d = 5$ should be the upper critical dimension above which $\alpha = \beta = 0$. Our numerical results do not support these conjectures. To undersand the problem of surface roughening in high dimensions we studied the random diode-resistor network on a Cayley tree. Our results indicate that the critical dimension is $d_c = \infty$.

## 2. Models

The models A, B and C we have studied are straightforward generalizations to $d$ dimensions of the $d = 1+1$ models defined in [13,14],[27] and [28]. In each model the porous media is simulated by a cubic lattice with certain fraction $p$ of randomly blocked cells which represent the inhomogeneities of the media (see Fig. 1). The horizontal cross section of the lattice is the $d-1$ dimensional hypercube of volume $L^{d-1}$ with periodic boundaries. Every lattice cell can be wet or dry. At $t = 0$, all lattice cells are dry except those with vertical coordinate $h \leq 0$. At every time step, we simultaneously examine all dry cells on the wet-dry interface and decide whether each of these cells should become wet on the next time step. The decision concerning each cell is taken according to the deterministic rules specific to each model. In analogy with spreading percolation [2,32], we define shell $t$ as the set of cells that become wet on time step $t$.

In Model A, each cell adjacent to the interface becomes wet if (i) it is unblocked or if (ii) it lies below the unblocked cell, adjacent to the interface.

In Model B, the cell becomes wet if (i) it is unblocked and its nearest neighbor from below is wet or (ii) if the height of the highest wet cell in one of the nearest neighboring columns is larger then the height of the cell under consideration.

In Model C, rule (i) is the same as in Model B, but rule (ii) is slightly different. The blocked cell with the coordinates $(x_1, x_2, ..., x_{d-1}, h)$ becomes wet if its nearest neighbor from below is wet and at least one of the cells with the coordinates $(x_1 + \Delta x_1, x_2 + \Delta x_2, ..., x_{d-1} + \Delta x_{d-1}, h + \Delta h)$ is wet, where the increments $\Delta x_i$ and $\Delta h$ obey the following constraints: $\Delta h \geq 0$, $|\Delta x_i| \leq 1$; if $\Delta h > 0$ then it is sufficient that $|\Delta x_i| \leq 1$; if $\Delta h = 0$ then $\Delta x_i = 0$ or $\Delta x_i = (-1)^{h+x_i}$. Model C is in fact equivalent to the diode-resistor network in which each blocked cell corresponds to a diode, and each unblocked cell to a resistor connecting a pair of the opposite vertices of the cell. Each diode or resistor has orientation $(\Delta x_1, \Delta x_2, ..., \Delta x_{d-1}, -1)$, where $\Delta x_i = -(-1)^{x_i+h}$ are determined by the coordinates of the cell (See Fig. 1c).

Model A is analogous to spreading percolation [rule (i)] with erosion of overhangs [rule (ii)]. In Models B and C, the height of each column at each time step can increase only by one cell. In Model A, the erosion of overhangs corresponds to much faster local growth. Also, in Models B and C, the maximal difference in heights of the two neighboring columns is 2, while in Model A, it can be any number.

Model A was suggested by Buldyrev et al. [13–15], while Model B was originally proposed by Tang and Leschhorn [27] and can be considered as the discretized version of the Langevin equation with quenched noise and is very close to the models defined by Csahók et al. [24] and Parisi [25]. Comparison of the time development of all three models is shown in Fig. 1 for the same $d = 1 + 1$ configuration of blocked cells. The growth of each model is stopped by the spanning paths of directed percolation of the blocked cells with different definitions of connectivity (shown as a red fence of blocked cells). In Model A the path has five choices (North, South, East, North-East and South-East), in Model B three choices (East, North-East, and South-East), in Model C only two choices (North-East and South-East). This results in different percolation thresholds: $p_c^A = 0.4698$, $p_c^B = 0.539$, $p_c^C = 0.6447$. Numbers on the cells show the time at which they become wet. These models have different finite size corrections to scaling, but the asymptotic behavior is characterized by the same set of exponents.

## 3. Dimension $1 + 1$: Theory and Simulations

In this section, we review theory and simulations for $d = 1 + 1$ [13-15]. When the probability of blocked cells $p$ is close to $p_c$, the growth is halted in many places by the paths of a directed percolation cluster. Each path can be characterized by two correlation lengths, $\xi_\perp$ and $\xi_\parallel$. When the path is spanning, i.e., when the growth is stopped completely, $\xi_\parallel$ is equal to the system size $L$, and $\xi_\perp$ is proportional to the width of the interface $w$. It is known from the theory of directed percolation [32,33] that the correlation lengths diverge in the vicinity of $p_c$ as

$$\xi_\perp \sim |p - p_c|^{-\nu_\perp}, \quad \xi_\parallel \sim |p - p_c|^{-\nu_\parallel}, \tag{3a}$$

where $\nu_\parallel \simeq 1.733$, and $\nu_\perp \simeq 1.097$ [32]. Thus,

$$w \sim \xi_\perp \sim L^{\nu_\perp / \nu_\parallel} \equiv L^\alpha, \tag{3b}$$

where

$$\alpha = \nu_\perp / \nu_\parallel \simeq 0.633 \pm 0.001. \qquad (3c).$$

Thus the spanning path of directed percolation describes the final state of the paper wetting experiment [13–15], where the wetting front is completely pinned by inhomogeneities in the paper. The theoretical value of $\alpha$ [Eq. (3c)] is in excellent agreement with both our simulations and our experiments [13–15].

A natural question is "What is the *dynamics* of wetting?" To answer this, we study the dynamical behavior of the models below and above $p_c$. Figure 2 shows a snapshot of the wetting front as it continues to propagate in the $(1+1)$-dimensional media when $p < p_c$. Large sections of the interface are already pinned and the growth is occurring only in columns that contain unblocked cells on the wet boundary (shown in red). The average horizontal size of pinned sections is $\xi_{\|}$, while the average vertical size of pinned sections is $\xi_{\perp}$. The moving parts have constant steep slope [in Model B it is exactly equal to 2], and their vertical and horizontal dimensions are proportional to $\xi_{\perp}$. The height-height correlation function $c(\ell, t)$, calculated for such an interface for $\ell \simeq \xi_{\perp}$ is of the order of $\ell^{(3\alpha-1)/2\alpha} = \ell^{0.71}$. When $\xi_{\perp}$ is very large, the effective value of the exponent $\alpha$ for $l < \xi_{\perp}$ is close to 1. The effective roughness exponent of this *moving* interface—calculated directly from our numerical data near criticality in $d = 1+1$—is $\alpha_{\rm dyn} = 0.70 \pm 0.05$, in good agreement with the above geometrical arguments. The growth is now mostly a fast erosion of steep slopes, propagating horizontally with constant speed. This observation implies that the dynamical exponent $z_{\rm dyn} \equiv \alpha_{\rm dyn}/\beta$ has a value close to one. Thus, $\beta \simeq \alpha_{\rm dyn}$, in good agreement with our numerical results [13–15]. This large value of $\alpha_{\rm dyn}$ may explain the large values $0.7 - 0.8$ found in dynamical experiments [11–12].

In the case $d = 2+1$ we find that for all studied models $\alpha = 0.48 \pm 0.02$, $\beta = 0.41 \pm 0.02$ and $z = 1.16 \pm 0.02$, while the results for $p_c$ for the different models are shown in Table 1. In $d > 2$ the duality to directed percolation breaks down, and the growth of the surface is pinned by self-affine hypersurfaces, that can be characterized by horizontal and vertical correlation lengths $\xi_{\|}$ and $\xi_{\perp}$ (see Ref. [34] for an isotropic model of percolating hypersurfaces).

Below criticality moving parts form steep circular terraces, surrounding pinned parts (see Fig. 4) . However, the moving parts in $d > 1+1$ do not move along straight lines, but rather perform a kind of correlated random walk, which implies $z \geq 1$. At $d = d_c$, $z$ should become 2, as for uncorrelated random walk. We find that even for $d = 5$ , $z = 1.5 \pm 0.1 < 2$, suggesting that $d_c > 5$.

The effective roughness exponent found for the moving interface in $d = 2+1$ near criticality is about $0.52 \pm 0.05$ which is in good agreement with our experiments on wetting of $3d$ porous media [19].

**4. Avalanches and Fractal Dust**

Above $p_c$, the growth is stopped by the spanning path of a directed percolation cluster in $d = 1+1$, or by a self-affine surface in $d = 2+1$. However, we can modify our models and assume that even when the growth is completely stopped, the blocked cells on the interface may still erode—but at an infinitesimal rate. With this assumption, we can remove blocked cells at random when the interface is completely stopped. Each removal will produce an avalanche of growth which eventually will die out when the front reaches another directed spanning path, or directed surface of blocked cells (see Fig. 3).

The distribution of avalanche sizes $P(V)$ is found to be [15]

$$P(V) \sim V^{-\tau_{\rm aval}} F\left(\frac{V}{V_0}\right), \tag{4a}$$

where $V$ is the number of sites removed in an avalanche, and $V_0 \sim \xi_{\|}^{d-1}\xi_{\perp}$ is the characteristic volume. The probability $P(V)$ is estimated to be the ratio of the number of avalanches of size $V$ to the total number of avalanches.

In $d = 1 + 1$, the maximum linear extent of the avalanches (Fig. 3) in the longitudinal and transverse directions is found to scale with exponents

$$\nu_{\|}^{\text{aval}} = 1.73 \pm 0.02, \qquad \nu_{\perp}^{\text{aval}} = 1.10 \pm 0.02 \tag{4b}$$

in excellent agreement with the correlation-length exponents of directed percolation. Moreover, we find [15,35]

$$\tau_{\text{aval}} = 1.26 \pm 0.03. \tag{4c}$$

This result is in agreement with $\tau_{\text{aval}} = 2/(1 + \alpha)$ found for $d = 1 + 1$ by Olami et al [36].

¿From avalanche studies in $d = 2 + 1$ we find the correlation-length and roughness exponents to be $\nu_{\|} = 1.06 \pm 0.1$, $\nu_{\perp} = 0.47 \pm 0.1$, which gives slightly smaller value of $\alpha$ than from analysis of height-height correlation function and width: $\alpha = \nu_{\perp}/\nu_{\|} = 0.44 \pm 0.1$. This may be due to large error bars that are caused by comparatively small errors in the value of $p_c$.

An alternative way of producing avalanches is to start growth from a single unblocked cell at time $t = 0$ when interface is flat (Fig. 4). Above $p_c$, the clusters of wet cells will be all pinned by the blocked cells. Below $p_c$ some of these clusters will grow infinitely, but some will be stopped by the pinning surfaces.

In analogy with conventional percolation, the survival probability $P_{\text{surv}}(t)$ of the clusters for $t < t_o$ will decay as a power law, $P_{\text{surv}}(t) \sim t^{1-\tau_{\text{surv}}}$. Here $t_o$ is the characteristic time which is related to the correlation lengths: $\xi_{\|} \sim t_o^{1/z}$, $\xi_{\perp} \sim t_o^{\beta}$. The exponent $\tau_{\text{surv}}$ is related to the $\tau_{\text{aval}}$ of Eq. (4):

$$(\tau_{\text{surv}} - 1) = (\tau_{\text{aval}} - 1)(d - 1 + \alpha)/z. \tag{5a}$$

For $t > t_o$, $P_{\text{surv}}(t)$ either goes to zero exponentially for $p > p_c$ or approaches a constant value $P(\infty)$, the probability of an infinite cluster, for $p < p_c$. Thus, studying $P_{\text{surv}}(t)$ provides very accurate method of estimating $p_c$. The critical exponents can be derived from the parameters of the clusters exactly at $p_c$. One of these exponents is the exponent $\delta$ that characterizes the time dependence of the size of the percolation shell: $n(t) \sim t^{\delta}$. It is possible to connect this exponent with $z$ and $\alpha$:

$$\delta + 1 = (d - 1 + \alpha)/z. \tag{5b}$$

In $d = 1+1$, $z = 1$, so $\delta = \alpha = \beta$, which agrees with the simple geometrical picture (Fig. 2) that the projection of the shell corresponds to the length of the largest steep moving terrace, which scales as the vertical size of the whole system $w(t) \sim t^{\beta}$. We have studied the time dependence of the size of the shell $n(t) \sim t^{\delta}$ numerically. The results are in agreement with the above relations.

The projection of the shell forms a *fractal dust* (Figs. 2 and 4). In $d = 1 + 1$, when $z = 1$, the fractal dimension of this dust $d_f$ is connected to the exponent of distribution of avalanches (i. e. blocked regions) through the relation

$$d_f = \tau_{\text{surv}} - 1 = 0.46 \pm 0.02. \tag{6}$$

Note that $d_f < \delta$ which means that, in fact, fractal dust is packed in moving blocks (the largest moving block is about of the vertical system size $w(t)$). These moving blocks behave like quasi-particles which are distributed in a fractal way with fractal dimension equal to $d_f$. Direct numerical studies of the correlation function of the dust support this point of view.

## 5. Possible generalization of the diode-resistor problem to the $d \to \infty$ limit

The generalization of the random diode-resistor problem to infinite dimension ($d \to \infty$) is next considered. If our assumptions are correct, the model we study corresponds to the Cayley tree limit [32,37].

Consider a Cayley tree with coordination number $z_c = 4$ [31] (Fig. 5). We assign an altitude $i \geq 0$ for each site on the Cayley tree. In this way, each site at level $i$ is connected via two bonds to a higher level $i+1$, and by two bonds to a lower level $i-1$. The sites on the level $i=0$ are connected only to the sites at level $i=1$.

Next, with probability $p$ a bond will be occupied by diode pointing to a lower level or with probability $1-p$ by a resistor. There exists a diode-resistor percolation threshold, $p = p_c$, above which there is no current between a site at level $i=0$ and a site at level $i=+\infty$. Below $p_c$ there is a finite probability that a site at level $i=0$ will be connected with $+\infty$ by a path of resistors and diodes on which current can flow from $i=0$ to $i=+\infty$. We define a cluster with respect to a given site at level $i$ as the set of sites that can be reached by a current starting at this site. We also define a cluster brunch to be a set of sites that are connected to a given site through one of the adjacent bonds.

To find $p_c$ and the critical behavior we define the following quantities. Let $F_i(s)$ be the probability that a site at level $i$ is connected via an upper bond to a branch of size $s$ and $H_i(s)$ is the probability that a site at level $i$ is connected via a lower bond to a branch of size $s$. Solving recursion relations for the corresponding generating functions, we find that the probability to be connected to the infinite cluster has the novel form [38]

$$P_\infty \sim B \exp\left(-\frac{A}{\sqrt{p_c - p}}\right), \tag{7}$$

where $A = (\pi/3)\sqrt{2}\ln 3$ for $z = 4$. From Eq. (7) follows that the exponent $\beta_p$ defined by $P_\infty \sim (p_c - p)^{\beta_p}$ has the value $\beta_p = \infty$ for the diode resistor on a Cayley tree.

We find the exact value of the critical threshold to be $p_c = 1 - 1/(z_c - 1)^2$. Furthermore, we find that the correlation function (the probability that a site on the level 0 is connected to a site on level $h$) decays at the critical point exponentially as $(z_c - 1)^h$, which means that $\xi_\perp$ is finite and hence $\nu_\perp = 0$.

The logarithmic derivatives of the generating functions give the average sizes of branches that are connected to a site on level $i$ from above and below respectively. Solving the corresponding recursion equations, we obtain that average size of the finite branch is finite below and above $p_c$, but has a first order discontinuity at $p = p_c$. This implies that $\gamma = 0$. We also find that the average cluster size, that reached level $h$, scales like $(z_c - 1)^h$, while the minimal cluster size at the level $h$ scales like $(z_c/2)^h$. This means that the average number of summits of such a cluster scales like $(2 - 2/z_c)^h$. These summits perform a random walk in the horizontal dimension of the Cayley tree, which means that the horizontal radius of the cluster is the square root of the number of summits, which still grows exponentially with $h$, hence the roughness exponent is $\alpha = 0$. We find, however, that the horizontal characteristic length $\xi_\parallel$ remains finite at the critical point, hence $\nu_\parallel = 0$ [38].

Direct studies of the cluster size distribution obtained show that the exponent of the cluster size distribution $\tau_{\rm aval}$ is a monotonically increasing function of the diode probability $p$. Exactly at the critical point $F_o(s) = (s \ln s)^{-2}[C + O(1/\ln s)]$, which is consistent with the first order discontinuity of the average cluster size observed at the critical point. In the standard percolation notations, this means that $\tau = 3$, larger than $\tau = 5/2$ for the regular Cayley tree. Below the critical point $\tau_{\rm aval}(p) < 2$, but there appears an abrupt exponential cutoff of the distribution at the cluster size $s_o \sim \exp(A/\sqrt{p_c - p})$, implying $\sigma = 0$. The derivation of the above results will be published elsewhere [38].

Using hyperscaling equation with the above values of the exponents $\nu_\perp + \nu_\parallel(d-1) = \gamma + 2\beta_p$ we suggest that the critical dimension of diode-resistor problem is $d_c = \infty$.

## 6. Discussion

We have studied several models of surface growth in the quenched disordered media near the pinning threshold. These models are in the universality class of the diode-resistor reverse percolation. The numerical results are in good agreement with anomalously large values of the critical exponents, obtained in many experiments in $d = 1+1$ and $d = 2+1$. The importance of quenched noise and pinning as a mechanism of surface roughening was suggested by several authors [24-26,39] which studied the continuum Langevin equations with quenched noise as models of surface growth. Our numerical results are in rather good agreement with the numerical results of those authors in $d = 1+1$, but differ in prediction of the behavior in high dimensions.

The behavior of the model on the Cayley tree suggests that the upper critical dimension is equal to infinity, in disagreement with [24,25]. It should be mentioned, that the invasion variant of the present models [14] ,where at time each step a site with the smallest resistivity is taken out, was later interpreted as the model of self-organized criticality [36,40–41]. However, all the properties of the later model, including the distribution of avalanches, can be derived from the corresponding properties of the present models.

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**Fig. 1:** Explanation of the surface growth models A, B and C in d=1+1. Cells are randomly blocked with probability $p$ (shown in blue) or unblocked with probability $1-p$ (shown in green). Wet cells are shown in white (those that were unblocked before the become wet) or yellow (those that were blocked). Wet cells are marked by numbers indicating time step at which they have become wet. Those blocked cells, that form the spanning path of directed percolation that has stopped the growth are shown in red. (a): Model A, proposed by Buldyrev et al. [13-15] ($p_c = 0.4698$), (b): Model B proposed by Tang and Leschhorn [27] ($p_c = 0.5390$), (c): Model C proposed by Dhar et al. [28] ($p_c = 0.6447$). The configuration of blocked cells is the same for all three models, so the largest cluster of model C corresponds to the largest $p_c$. In model C diodes correspond to blocked cells resistors to the unblocked cells. The order of wetting differs from the original definition of Ref. [28] in order to prevent overhangs in the moving parts of the interface. These changes, however, does not affect the the shape of the pinned interface, which in both cases is the path of the bond directed percolation of diodes on the dual lattice (shown on Fig.

**Fig. 2:** Two successive snapshots of the interface of Model A still evolving near its pinning threshold (p=0.4698). System size is $2^{12}$, but the calculation are made for $L \leq 2^{17}$. Upper snapshot taken at $t = 18000$, lower at $t = 19500$. Yellow color indicates wet area, blue color indicates dry area, red color indicates "live" columns, i. e. columns that contain cells which become wet at the current time step. Arrows show the direction of the propagation of steep eroding slopes, which has approximately equal horizontal and vertical dimensions both of the order of $\xi_\perp$. The horizontal dimension of the long blocked region is of the order of $\xi_\|$, while its vertical dimension is of the order of $\xi_\perp$.

**Fig. 3:** Successive series of pinned interfaces of Model A, showing the boundaries of avalanches, produced by removing a randomly-chosen blocked cell from the previously-pinned interface. $L = 400$, $p = 0.5 > p_c$. Correlation lengths $\xi_\|$ and $\xi_\perp$ are the typical sizes of the avalanches.

**Fig. 4:** Horizontal projection of the cluster of Model B ($p = 0.8009 = p_c$) which was started at the center of the screen $2^{10}$ time steps ago. The current diameter of the cluster is about $2^{10}$. Blue area shows flat interface that left dry since the beginning of the process. Darkest shades of gray corresponds to the largest heights of the interface. Red dots forming "fractal dust" indicate cells that become wet at the current time step.

**Fig. 5:** A finite diode-resistor cluster of size $s = 9$ on the Cayley tree with coordinational number $z_c = 4$. The initial site is marked by a larger circle. Sites are arranged vertically according to the number of the level they belong to. The lowest level has the number equal to zero. In order to illustrate the concept of a branch, we show several examples of branches of various sizes $s$ outgoing from sites at different levels $i$. These branches are indicated by arcs and probabilities $F_i(s)$, $H_i(s)$. For example, the right branch that is connected to the site at level $i = 2$ from below consist of $s = 5$ sites. Diodes that are connected to the cluster by their lower end only represent surface that stops percolation. These diodes are considered to be upper branches of zero mass, which always have probability $F_i(0) \equiv p$. Note that only 3 resistors (not 5 as shown) are necessary to create this cluster.

**Table 1: Critical Exponents and Percolation Thresholds $p_c^A$, $p_c^B$, $p_c^C$, for Models A,B, and C**

| $d$ | $1+1$ | $2+1$ | $3+1$ | $4+1$ |
|---|---|---|---|---|
| $\alpha$ | $0.63 \pm 0.01$ | $0.48 \pm 0.03$ | $0.38 \pm 0.04$ | $0.27 \pm 0.05$ |
| $\beta$ | $0.63 \pm 0.01$ | $0.41 \pm 0.02$ | $0.28 \pm 0.03$ | $0.18 \pm 0.03$ |
| $z$ | $1.01 \pm 0.02$ | $1.16 \pm 0.03$ | $1.32 \pm 0.5$ | $1.50 \pm 0.10$ |
| $\tau_{\text{surv}}$ | $1.46 \pm 0.02$ | $2.18 \pm 0.03$ | $2.54 \pm 0.05$ | $3.00 \pm 0.20$ |
| $\delta$ | $0.60 \pm 0.03$ | $1.14 \pm 0.06$ | $1.6 \pm 0.1$ | $1.9 \pm 0.2$ |
| $p_c^A$ | $0.4698 \pm 0.0002$ | $0.7423 \pm 0.0002$ | $0.8423 \pm 0.0002$ | $0.889 \pm 0.0005$ |
| $p_c^B$ | $0.5388 \pm 0.0002$ | $0.8009 \pm 0.0002$ | $0.8857 \pm 0.0004$ | $0.9240 \pm 0.0005$ |
| $p_c^C$ | $0.6447 \pm 0.0001$ | $0.9340 \pm 0.0003$ | $0.9863 \pm 0.0004$ | $0.9970 \pm 0.0005$ |