Expected number of distinct sites visited by N Lévy flights on a one-dimensional lattice

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We calculate asymptotic forms for the expected number of distinct sites, $\langle S_N(n)\rangle$, visited by N noninteracting n-step symmetric Lévy flights in one dimension. By a Lévy flight we mean one in which the probability of making a step of j sites is proportional to $1/|j|^{1+\alpha}$ in the limit $j\to\infty$. All values of $\alpha>0$ are considered. In our analysis each Lévy flight is initially at the origin and both N and n are assumed to be large. Different asymptotic results are obtained for different ranges in α. When n is fixed and $N\to\infty$ we find that $\langle S_N(n)\rangle$ is proportional to $(Nn^2)^{1/(1+\alpha)}$ for $\alpha<1$ and to $N^{1/(1+\alpha)}n^{1/\alpha}$ for $\alpha>1$. When α exceeds 2 the second moment is finite and one expects the results of Larralde *et al.* [Phys. Rev. A **45**, 7128 (1992)] to be valid. We give results for both fixed n and $N\to\infty$ and N fixed and $n\to\infty$. In the second case the analysis leads to the behavior predicted by Larralde *et al.* [S1063-651X(96)09705-X]

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I. INTRODUCTION

Although the problem of calculating properties of the number of distinct sites visited by an n-step lattice random walk, $S(n)$, was first suggested as being of purely mathematical interest [1], properties of this random variable have been extensively applied in a number of fields in the physical sciences [2–12]. For example, a knowledge of the behavior of $S(n)$ can be used to characterize the amount of territory reached by a diffusing particle. It is therefore useful for extending the Smoluchowski model for deriving macroscopic rate constants from a microscopic model of a chemical reaction [13,14].

Quite difficult mathematical problems arise in finding the probability distribution of $S(n)$. However, if attention is restricted to the first two moments of this random variable then a considerable amount of information can be learned about asymptotic properties because the generating functions for these quantities are known [15–17]. More sophisticated mathematical methods have also been used to find asymptotic properties of the second moment of $S(n)$ [18,19]. In principle, generating functions can be found for higher moments but the resulting analysis requires quite tedious calculations [17]. A knowledge of generating functions combined with the application of Tauberian methods enables one to calculate at least the first-order term in an asymptotic expansion of the moments.

The problem of finding moments of $S(n)$ as described in the preceding paragraphs has been analyzed only for a single random walker. More recently this analysis has been extended by Larralde *et al.* [20] to that of finding properties of the expected number of distinct sites visited by N noninteracting n-step random walkers, a quantity which will be denoted by $\langle S_N(n)\rangle$. Even in the simplest case of an isotropic random walk in which the single jump is bounded the behavior of $\langle S_N(n)\rangle$ was proven to be surprisingly rich when considered as a function of the two variables n and N. In the present work we calculate $\langle S_N(n)\rangle$ for random walkers in one dimension which have symmetric displacement probabilities having an asymptotically stable-law form.

Let $p(j)$ be the probability that any one of the random walkers makes a displacement equal to j in a single step. By the asymptotic stable-law form we will mean that in the limit $j\to\infty$, $p(j)$ has the property

$$p(j)\approx\frac{J^{\alpha}}{|j|^{1+\alpha}},\tag{1}$$

where J is a constant. Random walks having this property are special cases of what are generally termed Lévy flights [21–25], or, in mathematical terminology, are in the domain of attraction of stable laws [26]. Lévy flights were introduced as a class of random walks which have associated limit laws but may not have finite moments. They are fundamental in the discussion of non-Brownian enhanced diffusion. The asymptotic forms for $\langle S_1(n)\rangle$ for random walks characterized by the property in Eq. (1) was first derived by Gillis and Weiss [27]; see also [28,29].

When $\alpha\leqslant 2$ in Eq. (1) the second moment of displacement is infinite, leading to the expectation that the asymptotic behavior should differ from that found in [20]. On the other hand, when $\alpha>2$ the second moment is finite and thus one might expect that the results will be those obtained in [20]. However, we have found them to be correct in the $n\to\infty$ limit only, while for $N\to\infty$ the function $\langle S_N(n)\rangle$ differs from the results derived in [20].

II. DETAILS OF THE ANALYSIS

Let us begin by writing the formalism for calculating $\langle S_N(n)\rangle$ similar to that given in [20]. Let $p_n(j)$ be the probability that a single random walker is at site j at step n, and let $f_n(j)$ be the first-passage time probability for the random walker to be at j at step n. A function required for our

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analysis is the probability that the walker has *not* visited j by step n. This will be denoted by $\Gamma_n(j)$ which is related to the set of $f_m(j)$, $m=0,1,\ldots,n$, by

$$\Gamma_n(j)=1-\sum_{m=1}^{n} f_m(j). \tag{2}$$

The expected number of distinct sites visited by the N random walkers all starting at the same site is

$$\langle S_N(n)\rangle=\sum_j [1-\Gamma_n^N(j)], \tag{3}$$

where the sum is over all sites j.

When N is large, sites close to the origin tend to be visited after a small number of steps. Hence the principal contribution to $\langle S_N(n)\rangle$ at large n is dominated by the large-$|j|$ behavior. This allows us to simplify the analysis by requiring only a calculations the large-j form of $f_n(j)$. Let $\hat{p}(j;z)$ denote the generating function

$$\hat{p}(j;z)=\sum_{n=0}^{\infty} p_n(j)z^n \tag{4}$$

and $\hat{f}(j;z)$ be the analogous generating function for the $f_n(j)$. The relation between the two generating functions is

$$\hat{f}(j;z)=\hat{p}(j;z)/\hat{p}(0;z), \quad j\neq 0 \tag{5}$$

[28,30]. To find an approximate analytic form for $\hat{f}(j;z)$ valid for large $|j|$, and in the limit $z\to 1$ we can use the approximation to $p_n(j)$ valid at these values of j. These probabilities are readily shown to have the asymptotic form

$$p_n(j)\approx \frac{nJ^\alpha}{|j|^{1+\alpha}}. \tag{6}$$

In the indicated limits we can write for $\hat{p}(j;z)$

$$\hat{p}(j;z)\approx \frac{J^\alpha}{(1-z)^2|j|^{1+\alpha}}, \quad j\neq 0. \tag{7}$$

When $j=0$ we make use of the known integral representation of $\hat{p}(0;z)$ [28],

$$\hat{p}(0;z)=\frac{1}{\pi}\int_0^\pi \frac{d\theta}{1-z\hat{p}(\theta)}, \tag{8}$$

where $\hat{p}(\theta)=\Sigma_j p(j)\exp(ij\theta)$. The asymptotic property in Eq. (1) implies that in the neighborhood of $\theta=0$, $\hat{p}(\theta)$ can be expanded to lowest order as

$$\hat{p}(\theta)\approx\begin{cases}1-(J\theta)^\alpha, & \alpha\neq 2\\ 1-(J\theta)^2\ln(1/\theta), & \alpha=2\ .\end{cases} \tag{9}$$

In consequence, the behavior of $\hat{p}(0;z)$ in the $z\to 1$ limit is approximately

$$\hat{p}(0;z)\approx\begin{cases}\dfrac{1}{\pi}\displaystyle\int_0^\pi \frac{d\theta}{1-z+(J\theta)^\alpha}, & \alpha\neq 2\\ \dfrac{1}{\pi}\displaystyle\int_0^\pi \frac{d\theta}{1-z+(J\theta)^2\ln(1/\theta)}, & \alpha=2,\end{cases} \tag{10}$$

and equal to the constant $\hat{p}(0;1)$ when $\alpha<1$. When $\alpha>1$ the integral is singular at $\theta=0$ but not at $\theta=\infty$. Hence calculations are simplified by approximating to the singular behavior in that limit by setting the upper limit equal to ∞. The resulting integral can be evaluated exactly, yielding the result

$$\hat{p}(0;z)\approx\frac{\csc(\pi/\alpha)}{J\alpha}\frac{1}{(1-z)^{1-1/\alpha}}, \quad z\to 1,\ 1<\alpha<2. \tag{11}$$

When $\alpha=1$ the limit of integration in Eq. (10) cannot be extended to ∞ without introducing an extraneous singularity. However, the middle integral in Eq. (10) is trivial integrable and implies that

$$\hat{p}(0;z)\approx\frac{1}{\pi J}\ln\left(\frac{1}{1-z}\right), \quad z\to 1,\ \alpha=1. \tag{12}$$

When $\alpha=2$ a slightly more complicated calculation leads to the result

$$\hat{p}(0;z)\approx\frac{1}{2J(1-z)^{1/2}}\ln^{-1}\left(\frac{1}{1-z}\right), \quad z\to 1,\ \alpha=2.$$

III. THE CASE $\alpha>1$

In order to make use of the expression in Eq. (3) it is necessary to find the large-$|j|$ approximation to $f_n(j)$. The starting point for doing so is the representation of $\hat{f}(j;z)$ shown in Eq. (5) together with the estimates in Eqs. (7) and (11). These lead to the approximation, valid in the limit $z\to 1$,

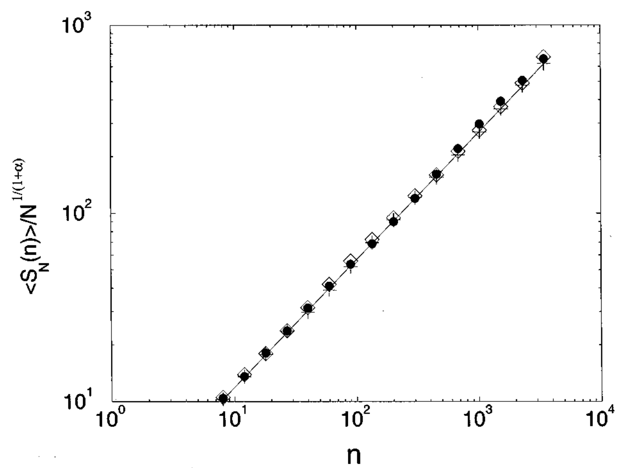

FIG. 1. Results obtained from 50 realizations of the case $\alpha=1.5$ compared with the prediction of Eq. (17). The fitted slope of the line is ≈ 0.67, which is to be compared to the theoretical value of $1/\alpha=\frac{2}{3}$. The plotted data corresponds to $N=100$ (●), 5000 (+), and 10 000 (◇).

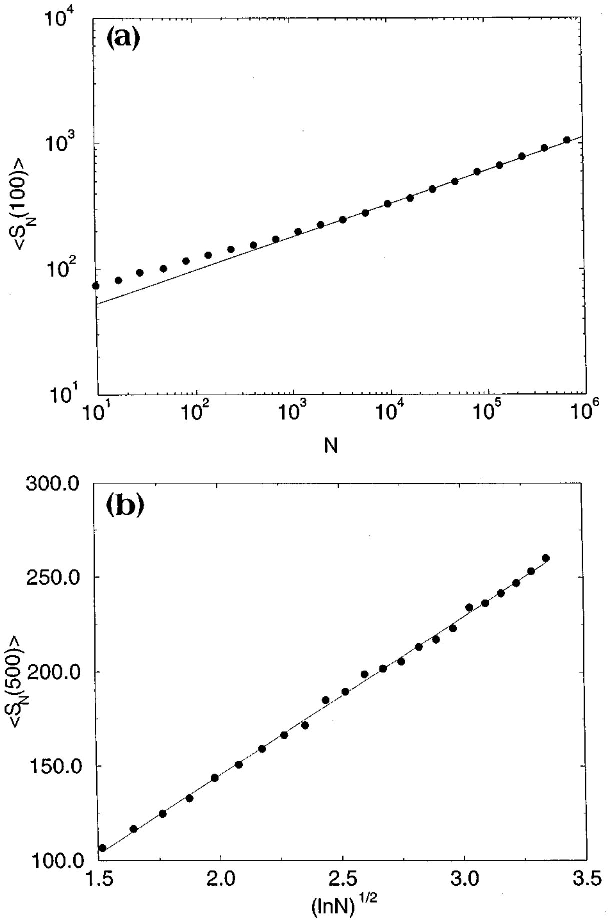

FIG. 2. (a) The crossover behavior for $\alpha=2.5$. For $n_\times=100$ and $N\approx 60$ the crossover occurs as predicted by Eq. (19). For $N=10^3 \gg 60$ the asymptotic slope is about 0.26, which is to be compared with the prediction $1/(1+\alpha)=0.286$. (b) A line fitted to simulated results for $\langle S_N(n)\rangle$ for $\alpha=5$ and $n=500$. These results correspond to the asymptotic behavior derived by Larralde *et al.* [20], before the crossover indicated by Eq. (19).

$$\hat{f}(j;z)\approx\begin{cases}\dfrac{J^{1+\alpha}\alpha\,\sin(\pi/\alpha)}{|j|^{1+\alpha}(1-z)^{1+1/\alpha}}, & \alpha\neq 2\\ \dfrac{2J^3}{|j|^3(1-z)^{3/2}}\ln^{-1}\left(\dfrac{1}{1-z}\right), & \alpha=2.\end{cases} \tag{13}$$

The use of a Tauberian theorem can be invoked to yield the asymptotic n-dependent behavior

$$f_n(j)\approx\frac{J^{1+\alpha}\sin(\pi/a)}{|j|^{1+\alpha}\Gamma(1/\alpha)}\,n^{1/\alpha}=K\,\frac{n^{1/\alpha}}{|j|^{1+\alpha}},\quad \alpha\neq 2\ ,$$
$$\approx K\,\frac{n^{1/2}}{|j|^3\ln n},\quad \alpha=2, \tag{14}$$

where K is the numerical coefficient indicated in the detailed expression. We next return to Eq. (2), which, for large values of n, can be approximated by replacing the sum by an integral, thus yielding

$$\Gamma_n(j)\approx 1-\int_0^n f_m(j)dm\approx\begin{cases}1-K'\,\dfrac{n^{1+1/\alpha}}{|j|^{1+\alpha}}, & \alpha\neq 2\\ 1-\dfrac{2K}{3}\,\dfrac{n^{3/2}}{|j|^3\ln n}, & \alpha=2,\end{cases} \tag{15}$$

where $K'=\alpha K/(1+\alpha)$. This approximation will be valid for values of n that satisfy $|j|\gg n^{1/\alpha}$. Since this means that the second term on the right-hand side of Eq. (15) is small in comparison to 1 we can derive a lowest-order approximation to $\langle S_N(n)\rangle$ by writing

$$\Gamma_n(j)\approx\begin{cases}\exp\left(-K'\,\dfrac{n^{1+1/\alpha}}{|j|^{1+\alpha}}\right), & \alpha\neq 2\\ \exp\left(-\dfrac{2K}{3}\,\dfrac{n^{3/2}}{|j|^3\ln n}\right), & \alpha=2,\end{cases} \tag{16}$$

and

$$\begin{aligned}\langle S_N(n)\rangle &\approx 2\int_0^\infty[1-\Gamma_n^N(j)]dj\\ &\approx 2\int_0^\infty\left[1-\exp\left(-K'\,\frac{Nn^{1+1/\alpha}}{j^{1+\alpha}}\right)\right]dj\\ &=2\Gamma\left(\frac{1}{1+\alpha}\right)(K'N)^{1/(1+\alpha)}n^{1/\alpha},\quad \alpha\neq 2\\ &\approx 2\Gamma\left(\frac{1}{3}\right)\left(\frac{2N}{3}\right)^{1/3}\frac{n^{1/2}}{\ln n},\quad \alpha=2.\end{aligned} \tag{17}$$

The prediction in Eq. (17) is compared with simulated data for $\alpha=1.5$ in Fig. 1. It should be noted that for $\alpha>2$ the second moment of the flights distribution, Eq. (1), becomes finite and the approximation (6) is valid for $j\gg n^{1/\alpha}$. Changing of the lower integration limit in (17) from zero to $n^{1/\alpha}$ affects only the constant in (17). As for $j\ll n^{1/\alpha}$, the probabil-

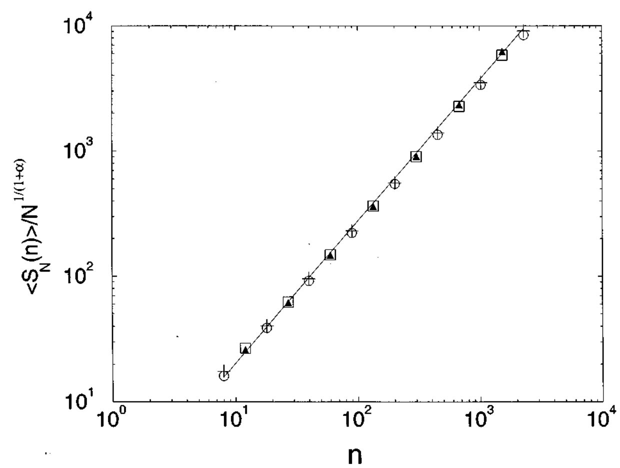

FIG. 3. A line fitted to data obtained from 50 realizations according to Eq. (24) for $\alpha=0.75$. The slope of the line is close to 1.13, which is in agreement with the theoretical value of $2/(1+\alpha)\approx 1.143$. The values of N presented are 1000 (○), 5000 (+), 10 000 (□), and 50 000 (△).

TABLE I. Asymptotic results obtained in the present work for different regimes of α.

	$\alpha<1$	$\alpha=1$	$1<\alpha<2$	$\alpha=2$	$\alpha>2$
$N\to\infty$	$(Nn^2)^{1/(1+\alpha)}$	$N^{1/2}n(\ln n)^{-1/2}$	$N^{1/(1+\alpha)}n^{1/\alpha}$	$N^{1/3}n^{1/2}$	$N^{1/(1+\alpha)}n^{1/\alpha}$
$n\to\infty$	Nn	$N^{1/2}n(\ln n)^{-1/2}$	$N^{1/(1+\alpha)}n^{1/\alpha}$	$N^{1/3}n^{1/2}(\ln n)^{-1}$	$(n\ \ln N)^{1/2}$

ity $p_n(j)$ can be approximated by a Gaussian and the corresponding results have been obtained by Larralde *et al.* [20]:

$$\langle S_N(n)\rangle \propto [n\ \ln(N)]^{1/2}. \tag{18}$$

Therefore, for $\alpha>2$ we have for $\langle S_N(n)\rangle$ the sum of the results of Eqs. (17) and (18). One can see that in the $n\to\infty$ limit the highest-order term is that from Eq. (18) and in the $N\to\infty$ limit the highest-order term is that from Eq. (17).

In comparing the dependence of $\langle S_N(n)\rangle$ on n and N in simulated data we found that indeed for comparatively small values of n the data agrees with the prediction of Eq. (17), as can be seen from the plot in Fig. 2(a) for $\alpha=2.5$. A crossover to the behavior predicted in Eq. (18) occurs at later times. Figure 2(b) shows simulated data in the region in which Eq. (18) holds, which corresponds to a relatively small number of random walkers. The behavior indicated in Eq. (17) occurs as the number of random walkers increases. The time at which the crossover occurs, $n_\times$, is found by equating these two equations and is found to satisfy

$$n_\times \propto \left(\frac{N^{2/(1+\alpha)}}{\ln(N)}\right)^{\alpha/(\alpha-2)} \tag{19}$$

so that as $\alpha\to 2+$ the value of $n_\times$ tends to infinity, which means that the regime in Eq. (18) no longer exists when $\alpha<2$.

IV. THE CASE $\alpha=1$

In this case we find

$$\hat{f}(j;z)\approx \frac{\pi J^2}{(1-z)^2 j^2 \ln\left(\frac{1}{1-z}\right)}. \tag{20}$$

Since the logarithm is a slowly varying function we can infer from this that in the limits $n\to\infty$ and $j^2\gg n^2$

$$\Gamma_n(j)\approx 1-\frac{\pi J^2 n^2}{2j^2\ln(n)}. \tag{21}$$

The analog of Eq. (17) then yields

$$\langle S_N(n)\rangle \approx \frac{2^{1/2}\pi J N^{1/2} n}{[\ln(n)]^{1/2}}. \tag{22}$$

Notice that the denominator is $\ln^{1/2}(n)$, in contrast to the first power of the logarithm that occurs for $N=1$ [27].

V. THE CASE $\alpha<1$

The same technique as used in the preceding sections can be used to calculate the form of $\langle S_N(n)\rangle$ when $\alpha<1$. In the present case we have

$$\hat{f}(j;z)\approx \frac{J^\alpha}{|j|^{1+\alpha}\hat{p}(0;1)(1-z)^2}, \tag{23}$$

which leads to the result

$$\langle S_N(n)\rangle \approx \left(\frac{2^\alpha}{\hat{p}(0;1)}\right)^{1/(1+\alpha)} \Gamma\left(\frac{1}{1+\alpha}\right)(Nn^2)^{1/(1+\alpha)}. \tag{24}$$

Behavior consistent with this prediction is illustrated by the data in Fig. 3.

Notice that Eq. (24) cannot hold in the limit of N fixed and $n\to\infty$, since for any value of α, it is obvious that $\langle S_N(n)\rangle$ must be less than Nn while the power to which n is raised in Eq. (24) is greater than 1. We conjecture that when N is fixed and $n\to\infty$ $\langle S_N(n)\rangle$ is actually proportional to nN, since at sufficiently long times the random walkers tend to separate, thereafter moving with minimal overlap. Consistent with this conjectured behavior would be a crossover time obtained from equating Eq. (24) with nN, which predicts that this occurs for values of n that satisfy

$$n \geqslant O(N^{\alpha/(1-\alpha)}). \tag{25}$$

VI. SUMMARY

We have found the asymptotic results for $\langle S_N(n)\rangle$, which are different for different ranges in α. All the results are displayed in Table I.

It has been shown that for $\alpha<1$ and for $\alpha>2$, the forms taken by $\langle S_N(n)\rangle$ depend on the order in which limits are taken: n fixed, $N\to\infty$ and N fixed, $n\to\infty$. By equating $\langle S_N(n)\rangle$ in the two regimes we can estimate the crossover time $n_\times$ for transitions between the two regimes. We find the crossover time between those regimes:

$$n_\times \propto \begin{cases} O(N^{\alpha/(1-\alpha)}) & \text{for } \alpha<1 \\ \left(\dfrac{N^{2/(1+\alpha)}}{\ln(N)}\right)^{\alpha/(\alpha-2)} & \text{for } 1<\alpha\leqslant 2. \end{cases}$$

Note, that when $\alpha\to 2+$ and $n_\times\to\infty$ only one regime exists.

It is interesting to note that the result for bounded step sizes derived in [20] is not valid in the limit $N\to\infty$. As seen in Table I one obtains the bounded step result [20] only when N is fixed and $n\to\infty$.

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