3 Directed and Non-directed Scale-Free Networks

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Abstract. Scale-free networks are networks with a scale-free degree distribution, *i.e.*, where the distribution of the number of links per node is a power-law, $p(k) = ck^{-\lambda}$. We review results for the properties of such networks, emphasizing the structural properties of these networks. We begin with normal random scale-free networks and present their percolation properties. We also review results for directed scale-free networks and their percolation properties. Finally we present a study of the possibility of embedding scale-free networks in a lattice.

3.1 Random Scale-Free Networks

The study of random network models began with Erdős and Rényi [1, 2, 3]. They studied models of networks with randomly distributed links. Those models lead to Poisson degree distributions [4]. Due to the development of computers, allowing the analysis of large amounts of data, and the formation of large scale networks, such as the Internet and WWW, some analysis of real world networks has been done in the last decade [5, 6, 7, 8, 9]. This research lead to the conclusion that real world networks are not described correctly by the ER model. The main difference found was that the degree distribution of real world networks studied was found to be very broad rather than the narrow Poisson distribution. Many of the networks studied can be fitted with a scale-free degree distribution. In this chapter we will elaborate on the properties of scale free networks.

A scale free network is a network having a degree distribution:

$$P(k) = ck^{-\lambda}, \tag{3.1}$$

where λ is the exponent and c is an appropriate normalization factor. The distribution is limited by the lower and upper cutoffs, which we will denote by m and K, respectively. The unique properties of this distribution stem from the fact that all moments with $n \geq \lambda - 1$ diverge with K (which is usually increasing with the size of the network).

3.1.1 Percolation Threshold

Percolation theory deals with the cluster structure of networks when a fraction of the sites or bonds is removed. A spanning cluster (or a "giant component"

R. Pastor-Satorras, M. Rubi, A. Diaz-Guilera (Eds.): LNP 625, pp. 23–45, 2003.

in the terminology of random graphs) is a cluster of connected sites (*i.e.* where there is a path from each site to each other) of the order of the size of the entire network. Most standard treatments of percolation deal with lattices and regular graphs. However, a similar treatment can be applied to random networks.

For a general random network having degree distribution $P(k)$ to have a spanning cluster, a site which is reached by following a link from this cluster must have at least one other link on average to allow the cluster to exist. For this to happen the average degree of a site must be at least 2 (one incoming and one outgoing link) given that the site i is connected to j:

$$\langle k_i | i \leftrightarrow j \rangle = \sum_{k_i} k_i P(k_i | i \leftrightarrow j) = 2. \tag{3.2}$$

Using Bayes rule we get

$$P(k_i | i \leftrightarrow j) = P(k_i, i \leftrightarrow j)/P(i \leftrightarrow j) = P(i \leftrightarrow j | k_i) P(k_i)/P(i \leftrightarrow j), \tag{3.3}$$

where $P(k_i, i \leftrightarrow j)$ is the *joint* probability that node i has degree k_i and that it is connected to node j. For randomly connected networks (neglecting loops) $P(i \leftrightarrow j) = \langle k \rangle/(N-1)$ and $P(i \leftrightarrow j | k_i) = k_i/(N-1)$, where N is the total number of nodes in the network. Using the above criteria (3.2) reduces to [10, 11]:

$$\kappa \equiv \frac{\langle k^2 \rangle}{\langle k \rangle} = 2, \tag{3.4}$$

at the critical point. A spanning cluster exists for graphs with $\kappa > 2$, while graphs with $\kappa < 2$ contain only small clusters whose size is not proportional to that of the entire network. This criterion was derived earlier by Molloy and Reed [10] using a somewhat different arguments.

The negligence of loops can be justified below the threshold since the probability for a bond to form a loop in an s-node cluster is proportional to $(s/N)^2$ (i.e., proportional to the probability of choosing two sites in that cluster). Calculating the fraction of loops P_{loop} in the system yields:

$$P_{loop} \propto \sum_i \frac{s_i^2}{N^2} < \sum_i \frac{s_i S}{N^2} = \frac{S}{N}, \tag{3.5}$$

where the sum is over all clusters in the system and s_i is the size of the ith cluster [12]. Therefore, the fraction of loops in the system is less than or proportional to S/N, where S is the size of the largest cluster. Below the critical threshold there is no spanning cluster in the system and therefore the fraction of loops is negligible. Hence, for values of κ below $\kappa = 2$, loops can be neglected. At the threshold the structure of the spanning cluster is almost a tree. Above the threshold loops can no longer be neglected, but since this only happens when a spanning cluster exists the criterion in (3.4) is valid as a criterion for finding the critical point. A derivation of the exact conditions under which (3.4) is valid can be found in [10].

The above reasoning can be applied to the problem of percolation on a generalized random network. If we randomly remove a fraction p of the sites (or bonds), the degree distribution of the remaining sites will change. For instance, sites with initial degree k_0 will have, after the random removal of nodes, a different number of connections, depending on the number of removed neighbors. The new number of connections will be binomially distributed. If we begin with a distribution of degrees $P_0(k_0)$, the new degree distribution of the network will be:

$$P(k) = \sum_{k_0=k}^{\infty} P_0(k_0) \binom{k_0}{k} (1-p)^k p^{k_0-k}. \tag{3.6}$$

Calculating the first moment for this distribution, given $\langle k_0 \rangle$ and $\langle k_0^2 \rangle$ for the original distribution leads to:

$$\langle k \rangle = \sum_{k=0}^{\infty} P(k)k = (1-p)\langle k_0 \rangle. \tag{3.7}$$

In the same manner we can calculate the second moment:

$$\langle k^2 \rangle = \sum_{k=0}^{\infty} P(k)k^2 = (1-p)^2\langle k_0^2 \rangle + p(1-p)\langle k_0 \rangle. \tag{3.8}$$

Both those quantities can be substituted into (3.4) to find the criterion for criticality. This yields:

$$\kappa \equiv \frac{\langle k^2 \rangle}{\langle k \rangle} = \frac{(1-p)^2\langle k_0^2 \rangle + p(1-p)\langle k_0 \rangle}{(1-p)\langle k_0 \rangle} = 2. \tag{3.9}$$

Reorganizing (3.9), one gets the critical threshold for percolation [11]:

$$1 - p_c = \frac{1}{\kappa_0 - 1}, \tag{3.10}$$

where $\kappa_0 \equiv \langle k_0^2 \rangle / \langle k_0 \rangle$ is calculated using the original distribution, before the removal of sites.

Eqations (3.4) and (3.10) are valid for a wide range of generalized random graphs and distributions. For example for a Cayley tree – a graph with a fixed degree z and no loops – the criterion from (3.10) can be used. This yields the critical concentration $q_c = 1 - p_c = 1/(z-1)$, which is well known [13]. Another example is a random Erdös-Rényi (ER) graph. In those graphs edges are distributed randomly and the resulting degree distribution is Poissonian [4]. Applying the criterion from (3.4) to a Poisson distribution yields:

$$\kappa \equiv \frac{\langle k^2 \rangle}{\langle k \rangle} = \frac{\langle k \rangle^2 + \langle k \rangle}{\langle k \rangle} = 2, \tag{3.11}$$

which reduces to $\langle k \rangle = 1$ as known for ER graphs [4].

Our main concern in this chapter will be with the behavior of scale-free networks. Scale-Free networks are networks whose degree distribution (i.e. fraction of sites with k connections) behaves as:

$$P(k) \propto k^{-\lambda},\ m \leq k \leq K, \tag{3.12}$$

where λ is the exponent, m is the lower cutoff, and K is the upper cutoff. There are no sites with degree below m and above K. For finite networks the upper cutoff N arises naturally since the fraction of high-degree sites decays with k. An estimate of this cutoff can be found by the assumption that the tail of the distribution above K is of the order of one site [11]:

$$\sum_{k=K}^{\infty} P(k) \sim \int_{K}^{\infty} P(k)dk = \frac{1}{N}. \tag{3.13}$$

The estimate obtained this way gives:

$$K \approx mN^{1/(\lambda-1)}. \tag{3.14}$$

This estimate allows the derivation of finite size effects in the network and allows calculations of moments of the distribution in (3.12), that would otherwise diverge. Newman *et al* [14] use an exponential cutoff rather than a sharp one, but the effect on the results is minor.

The importance of scale-free networks lies in the fact that this distribution occurs in many natural and man-made networks [5, 14, 15]. An example of a scale-free network is the physical Internet structure, that is the router to router (and end-units) connectivity. This structure was studied by Faloutsos *et al* [5]. They have found that the inter-router network is a non-directed scale-free network with $\lambda \approx 2.5$. The size of the Internet today is about 10^7 sites, making it a fairly large network.

Further results about the structure of scale-free networks have also been proven by Aiello *et al* [16]. The size of the infinite cluster was calculated, and it was found that for $\lambda \leq 2$ the infinite cluster is of almost the size of the entire graph (i.e. $P_\infty = 1 - o(1)$, where $o(1)$ is a function of the network size, $f(N)$, such that $f(N) \to 0$ when $N \to \infty$). For $\lambda > \lambda_c = 3.478\ldots$ there is no infinite cluster at all (since we use a somewhat different distribution [17], we get $\lambda_c \approx 4$). For $\lambda < \lambda_c$ the second largest cluster is of order lnN. For lower cutoff $m \geq 2$ a spanning cluster exists for every λ.

The average distance between sites is also different in scale free sites from its value for normal random graphs. While for ER graphs the average distance between sites behaves as $d \sim \ln N$ [4], for scale free graphs with $2 < \lambda < 3$ the distance behaves as $d \sim \ln \ln N$ [18, 19], for $\lambda = 2$, $d \sim const$, and for $\lambda = 3$, $d \sim \ln N / \ln \ln N$ [20]. The reason for this short distance is the small core, containing most high degree sites, which has a very small diameter. For $\lambda > 3$ the random graph behavior $d \sim \ln N$ is recovered. Those results were later confirmed using different methods in [21, 22].

3.1.2 Generating Functions

A general method for studying the size of the infinite cluster and the residual network for a graph with an arbitrary degree distribution was first developed by Molloy and Reed [23]. They suggested viewing the infinite cluster as being explored and used differential equations for the number of un-exposed links and unvisited sites to find the size of the infinite cluster and the degree distribution of the residual graph (the finite clusters).

An alternative and very powerful derivation was given by Newman, Strogatz and Watts [14]. They have used the generating functions method to study the size of the infinite cluster as well as other quantities (such as the diameter and cluster size distribution). They have also applied this method to other types of graphs (directed and bipartite). Here we closely follow their derivation in order to find the size of the infinite cluster and the critical exponents.

In [14, 24] a generating function is built for the degree distribution:

$$G_0(x) = \sum_{k=0}^{\infty} P(k)x^k. \tag{3.15}$$

The probability of reaching a site with degree k by following a specific link is $kP(k)/\langle k\rangle$ [10, 11, 14, 24], and the corresponding generating function for those probabilities is

$$G_1(x) = \frac{\sum kP(k)x^{k-1}}{\sum kP(k)} = \frac{d}{dx}G_0(x)/\langle k\rangle \ . \tag{3.16}$$

Assuming that $H_1(x)$ is the generating function for the probability of reaching a branch of a given size by following a link, the self-consistent equation for $H_1(x)$ is:

$$H_1(x) = 1 - q + qxG_1(H_1(x)) \ . \tag{3.17}$$

Since $G_0(x)$ is the generating function for the degree of a site, the generating function for the probability of a site to belong to an n-site cluster is:

$$H_0(x) = 1 - q + qxG_0(H_1(x)) \ . \tag{3.18}$$

Below the transition, $H_0(1) = 1$, since this is the probability to belong to a cluster of any size. However, above the transition this probability is no longer normalized since this does not include the infinite cluster. Then, the relative size of the giant cluster is $P_\infty = 1-q+qH_0(1)$, since H_0 contains only the finite-size clusters. It follows that

$$P_\infty = q\left(1 - \sum_{k=0}^{\infty} P(k)u^k\right) \ , \tag{3.19}$$

where $u \equiv H_1(1)$ is the smallest positive root (which can be found numerically) of

$$u = 1 - q + \frac{q}{\langle k \rangle} \sum_{k=0}^{\infty} kP(k)u^{k-1} \ . \tag{3.20}$$

This equation can be solved numerically and the solution can be substituted into (3.19) to calculate the size of the infinite cluster in a graph with a given degree distribution.

3.1.3 Critical Exponents

Using Abelian and Tauberian methods [25, 26] one can use . (3.19) and (3.20) to find the critical exponents for percolation in scale free networks. Some preliminary results can be found in [27]. A more detailed treatment can be found in [28, 19]. Here we just state the results.

The size of the giant component near the critical point behaves as $P_\infty \sim (p - p_c)^\beta$, where

$$\beta = \begin{cases} \frac{1}{3-\lambda} & 2 < \lambda < 3, \\ \frac{1}{\lambda-3} & 3 < \lambda < 4, \\ 1 & \lambda > 4. \end{cases} \tag{3.21}$$

The number of clusters with size s behaves as $n_s \sim (p - p_c)^{-\tau}$, where

$$\tau = 2 + \frac{1}{\lambda - 2} = \frac{2\lambda - 3}{\lambda - 2} \ , \qquad 2 < \lambda < 4 \ . \tag{3.22}$$

For $\lambda > 4$, $\tau = 2.5$, which is the regular mean field value. From those results it can be seen that the critical exponents are anomalous even when the second moment $\langle k^2 \rangle$ is convergent and only the third moment $\langle k^3 \rangle$ diverges, as in the case of $3 < \lambda < 4$.

From τ it can be deduced that the "double jump" in Erdős-Rényi graphs is also seen in scale free graphs, Where S, the size of largest component, scales as $S \sim N^{(\lambda-2)/(\lambda-1)}$ exactly at criticality [19]. For $\lambda \geq 4$ the known result of $S \sim N^{2/3}$ is obtained. The fractal dimensions at criticality for $\lambda > 3$ can also be obtained [19] and are:

$$d_l = \frac{\lambda - 2}{\lambda - 3}, \qquad d_f = 2\frac{\lambda - 2}{\lambda - 3}, \qquad d_c = 2\frac{\lambda - 1}{\lambda - 3}, \tag{3.23}$$

where for $\lambda \geq 4$ the regular mean field values of $2, 4, 6$ are restored.

3.2 Directed Graphs

Many complex networks in nature have directed links, a property that affects the network's navigability and large-scale topology. Here we study the percolation properties of such directed scale-free networks with correlated *in* and *out* degree

distributions. We derive a phase diagram that indicates the existence of three regimes, determined by the values of the degree exponents. In the first regime we regain the known directed percolation mean field exponents. In contrast, the second and third regimes are characterized by anomalous exponents, which we calculate analytically. In the third regime the network is resilient to random dilution, i.e., the percolation threshold is $p_c \to 1$.

Recently the topological properties of large complex networks such as the Internet, WWW, electric power grid, cellular and social networks have drawn considerable attention [29, 15]. Some of these networks are directed, for example, in social and economical networks [30] if node A gains information or acquires physical goods from node B, it does not necessarily mean that node B gets similar input from node A. Likewise, most metabolic reactions [31] are one-directional, thus changes in the concentration of molecule A affect the concentration of its product B, but the reverse is not true. Despite the directedness of many real networks, the modeling literature, with few notable exceptions [14, 32], has focused mainly on undirected networks.

An important property of directed networks can be captured by studying their degree distribution, $P(j,k)$, or the probability that an arbitrary node has j incoming and k outgoing edges. Many naturally occurring directed networks, such as the WWW, metabolic networks, citation networks, etc., exhibit a power-law, or *scale-free* degree distribution for the incoming or outgoing links:

$$P_{in(out)}(l) = cl^{-\lambda_{in(out)}}, \quad l \geq m \,, \tag{3.24}$$

where m is the minimal connectivity (usually taken to be $m = 1$), c is a normalization factor and $\lambda_{in(out)}$ are the in(out) degree exponents characterizing the network [6, 7]. An important property of scale-free networks is their robustness to random failures, coupled with an increased vulnerability to attacks [33, 11, 24, 27, 34]. Recently it has been recognized that this feature can be addressed analytically in quantitative terms [11, 24, 27] by combining graph theoretical concepts with ideas from percolation theory. Yet, while the percolation properties of undirected networks are much studied, little is known about the effect of node failure in directed networks. As many important networks are directed, it is important to fully understand the implications to their stability. Here we review and extend the results [35] showing that directedness has a strong impact on the percolation properties of complex networks and we draw a detailed phase diagram.

3.2.1 Structure

The structure of a directed graph has been characterized in [14, 32], and in the context of the WWW in [7]. In general, a directed graph consists of a giant weakly connected component (GWCC) and several finite components. In the GWCC every site is reachable from every other, provided that the links are treated as bi-directional. The GWCC is further divided into a giant strongly

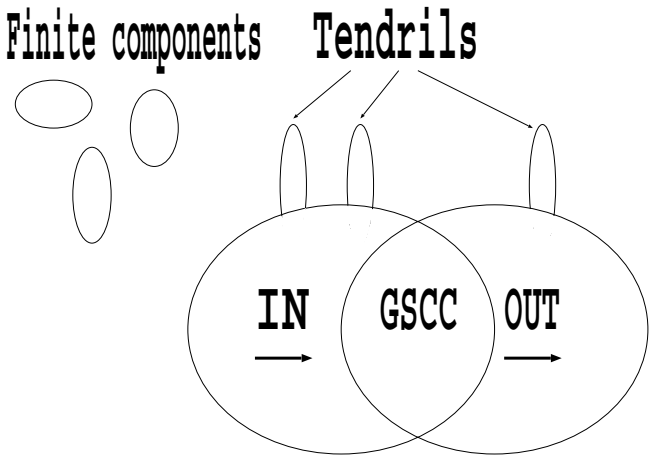

Fig. 3.1. Structure of a general directed graph

connected component (GSCC), consisting of all sites reachable from each other following directed links. All the sites reachable from the GSCC are referred to as the giant OUT component, and the sites from which the GSCC is reachable are referred to as the giant IN component. The GSCC is the intersection of the IN and OUT components. All sites in the GWCC, but not in the IN and OUT components are referred to as the "tendrils" (see Fig. 3.1).

3.2.2 Percolation Threshold

For a directed random network of arbitrary degree distribution the condition for the existence of a giant component can be deduced in a manner similar to [11]. If a site is reached following a link pointing to it, then it must have at least one outgoing link, on average, in order to be part of a giant component. This condition can be written as

$$\langle k_j | i \to j \rangle = \sum_{k_i,k_j} k_j P(k_i, k_j | i \leftrightarrow j) = 1. \tag{3.25}$$

Using Bayes rule we get

$$P(k_i, k_j | i \to j) = \frac{P(k_i, k_j, i \leftrightarrow j)}{P(i \to j)} = \frac{P(i \to j | k_i, k_j) P(k_i, k_j)}{P(i \to j)}. \tag{3.26}$$

For random networks $P(i \to j) = \langle k \rangle / (N-1)$ and $P(i \to j | k_i, k_j) = k_i/(N-1)$, where N is the total number of nodes in the network. The above criterion thus reduces to [14, 32]

$$\langle jk \rangle \geq \langle k \rangle. \tag{3.27}$$

Suppose a fraction p of the nodes is removed from the network. (Alternatively, a fraction $q = 1 - p$ of the nodes is retained.) The original degree distribution, $P(j,k)$, becomes

$$P'(j,k) = \sum_{j_0,k_0}^{\infty} P(j_0,k_0)\binom{j_0}{j}(1-p)^j p^{j_0-j}\binom{k_0}{k}(1-p)^k p^{k_0-k} \ . \tag{3.28}$$

In view of this new distribution, (3.27) yields the percolation threshold

$$q_c = 1 - p_c = \frac{\langle k\rangle}{\langle jk\rangle} \ , \tag{3.29}$$

where averages are computed with respect to the original distribution before dilution, $P(j,k)$. Equation (3.29) indicates that in directed scale-free networks if $\langle jk\rangle$ diverges then $q_c \to 0$ and the network is resilient to random breakdown of nodes and bonds.

The term $\langle jk\rangle$ may be dramatically influenced by the appearance of correlations between the *in*- and *out*-degrees of the nodes. In particular, let us consider scale-free distributions for both the *in*- and *out*-degrees:

$$P_{in}(j) \sim \begin{cases} Bc_{in}j^{-\lambda_{in}} & j \neq 0, \\ 1 - B & j = 0, \end{cases} \tag{3.30}$$

and

$$P_{out}(k) = c_{out}k^{-\lambda_{out}} \ . \tag{3.31}$$

In (3.30) we choose to add the possible zero value to the *in*-degree in order to maintain $\langle j\rangle = \langle k\rangle$. If the *in*- and *out*-degrees are uncorrelated, we expect $\langle jk\rangle = \langle j\rangle\langle k\rangle$. For several real directed networks this equality does not hold. For example, the network of Notre-Dame University WWW [6], has $\langle k\rangle = \langle j\rangle \approx 4.6$, and thus $\langle j\rangle\langle k\rangle = 21.16$. In contrast, measuring directly we find $\langle jk\rangle \approx 200$, about an order of magnitude larger than the result expected for the uncorrelated case. This yields an estimate of $q_c \approx 0.02$, i.e., a very stable directed network. Similar results are also obtained for metabolic networks studied in [31], indicating that in many real directed networks, the *in*- and *out*-degrees are correlated.

To address correlations, we model it in the following manner: we first generate the j values for the entire network. Next, for each site with $j \neq 0$ with probability A we generate k fully correlated with j, i.e., $k = k(j)$. Assuming that $k(j)$ is a monotonically increasing function then the requirement $c_{out}k^{-\lambda_{out}}dk = c_{in}j^{-\lambda_{in}}dj$ — needed to maintain the distributions scale-free — leads to $k^{\lambda_{out}-1} = j^{\lambda_{in}-1}$. With probability $1 - A$, the degree k is chosen independently from j:

$$P(j,k) \sim \begin{cases} (1-A)Bc_{in}j^{-\lambda_{in}}c_{out}k^{-\lambda_{out}} + BAc_{out}k^{-\lambda_{out}}\delta_{j,j(k)} & j \neq 0, \\ (1-B)c_{out}k^{-\lambda_{out}} & j = 0, \end{cases} \tag{3.32}$$

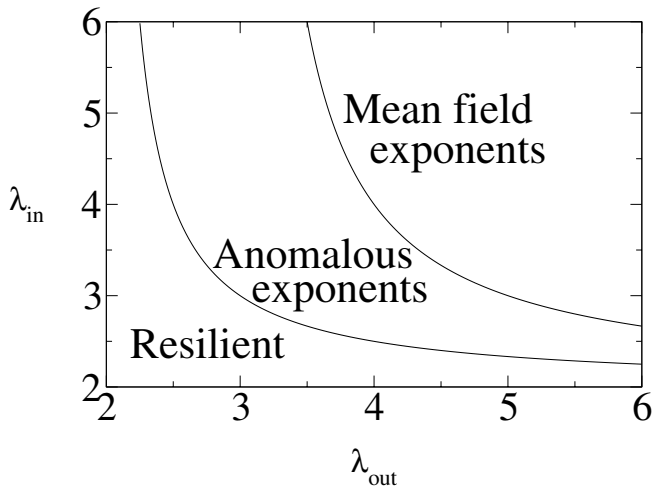

Fig. 3.2. Phase diagram of the different regimes for the IN component of scale-free correlated directed networks. The boundary between Resilient and Anomalous exponents is derived from (3.33) while that between Anomalous exponents and Mean field exponents is given by (3.48) for $\lambda^\star = 4$. For the diagram of the OUT component λ_{in} and λ_{out} change roles. After [35]

where $j(k) = k^{\frac{\lambda_{out}-1}{\lambda_{in}-1}}$. With this distribution, any finite fraction BA of fully correlated sites yields a diverging $\langle jk \rangle$ whenever

$$(\lambda_{out} - 2)(\lambda_{in} - 2) \leq 1 \,, \tag{3.33}$$

causing the percolation threshold to vanish (see Fig. 3.2). The influence of even very small correlation on the threshold, and the sharpness of the transition to the resilient regime can be seen in Fig. 3.3.

In the case of no correlations between the *in*- and the *out*-degrees, $A = 0$, (3.32) becomes $P(j,k) = P_{in}(j)P_{out}(k)$. Then the condition for the existence of a giant component is: $\langle k \rangle = \langle j \rangle = 1$. Moreover, (3.29) reduces to:

$$q_c = 1 - p_c = \frac{1}{\langle k \rangle} \,. \tag{3.34}$$

Applying (3.34) to scale-free networks one concludes that for $\lambda_{out} > 2$ and $\lambda_{in} > 2$ a phase transition exists at a finite q_c. Here we concern ourselves with the critical exponents associated with the percolation transition in both correlated and uncorrelated scale-free network of $\lambda_{out} > 2$ and $\lambda_{in} > 2$, which is the most relevant regime (Fig. 3.2).

Percolation of the GWCC can be seen to be similar to percolation in the non-directed graph created from the directed graph by ignoring the directionality of the links. The threshold is obtained from the criterion [11]

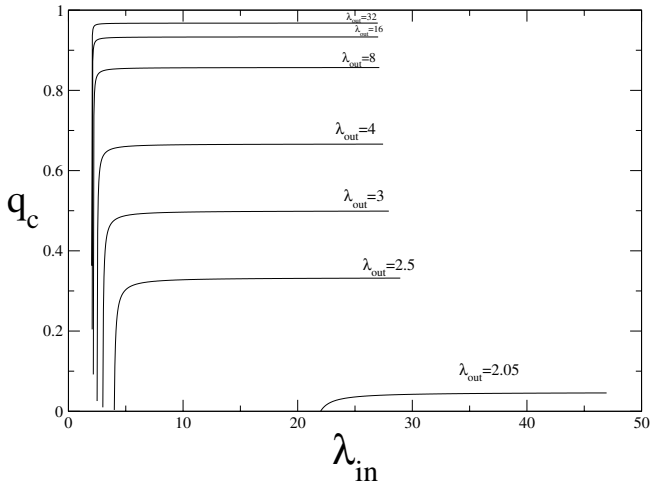

Fig. 3.3. The critical concentration as a function of λ_{in} and λ_{out}. Notice the steep change at the boundaries of the transition between the resilient and non resilient regimes. This plot was obtained for $A = 0.05$

$$q_c = \frac{\langle k \rangle}{\langle k(k-1) \rangle}. \tag{3.35}$$

Here the connectivity distribution is the convolution of the *in* and *out* distributions

$$P'(k) = \sum_{l=0}^{k} P(l, k-l). \tag{3.36}$$

Regardless of correlations, $P'(k)$ is always dominated by the slower decay-exponent, therefore percolation of the GWCC is the same as in non-directed scale-free networks, with $\lambda_{eff} = min(\lambda_{in}, \lambda_{out})$. Note that the percolation threshold of the GWCC may differ from that of the GSCC and the IN and OUT components [32].

3.2.3 Critical Exponents

We now use the formalism of generating functions [26] to analyze percolation of the GSCC and IN and OUT components [35]. In [14, 32] a generating function is built for the joint probability distribution of outgoing and incoming degrees, before dilution:

$$\Phi(x, y) = \sum_{k,j} P(j, k) x^j y^k \ . \tag{3.37}$$

Using the approach of Callaway *et al* [24], let $q(j,k)$ be the probability that a vertex of degree (j,k) remains in the network following dilution. The generating function after dilution is then

$$G(x,y) = \sum_{k,j} P(j,k) q(j,k) x^j y^k \ . \tag{3.38}$$

From (3.38) it is possible to define the generating function for the outgoing degrees G_0

$$G_0(y) \equiv G(1,y) = \sum_{k,j} P(j,k) q(j,k) y^k \ . \tag{3.39}$$

The probability of reaching a site by following a specific link is proportional to $jP(j,k)$, therefore, the probability to reach an occupied site following a specific directed link is generated by

$$G_1(y) = \frac{\sum_{j,k} jP(j,k) q(j,k) y^k}{\sum_{j,k} jP(j,k)} \ . \tag{3.40}$$

Let $H_1(y)$ be the generating function for the probability of reaching an outgoing component of a given size by following a directed link, after a dilution. $H_1(y)$ satisfies the self-consistent equation:

$$H_1(y) = 1 - G_1(1) + yG_1(H_1(y)) \ . \tag{3.41}$$

Since $G_0(y)$ is the generating function for the outgoing degree of a site, the generating function for the probability that n sites are reachable from a given site is

$$H_0(y) = 1 - G_0(1) + yG_0(H_1(y)) \ . \tag{3.42}$$

For the case where correlations exist, and assuming random dilution: $q(j,k) = q$, (3.41) and (3.42) reduce to

$$H_1(y) = 1 - q + \frac{qy}{\langle j \rangle} \sum_k (BAj(k) + (1-A)\langle j \rangle) P_{out}(k) H_1(y)^k \ , \tag{3.43}$$

and

$$H_0(y) = 1 - q + qy \sum_k P_{out}(k) H_1(y)^k \ . \tag{3.44}$$

If $A \to 0$, one expects that $H_0(y) = H_1(y)$, since there is no correlation between j and k, thus the probability to have k outgoing edges is $P_{out}(k)$ whether we choose the site randomly or weighted by the incoming edges j.

$H_0(1)$ is the probability to reach an outgoing component of any *finite* size choosing a site. Thus, below the percolation transition $H_0(1) = 1$, while above

the transition there is a finite probability to follow a directed link to a site which is a root of an infinite outgoing component: $P_\infty = 1 - H_0(1)$. It follows that

$$P_\infty(q) = q(1 - \sum_k^\infty P_{out}(k)u^k) \ , \tag{3.45}$$

where $u \equiv H_1(1)$ is the smallest positive root of

$$u = 1 - q + \frac{q}{\langle j \rangle} \sum_k (BAj(k) + (1 - A)\langle j \rangle) P_{out}(k)u^k \ . \tag{3.46}$$

Here $P_\infty(q)$ is the fraction of sites from which an infinite number of sites is reachable. Equation (3.46) can be solved numerically and the solution may be substituted into (3.45), yielding the size of the IN component at dilution $p = 1-q$.

Giant Component Size

Near criticality, the probability to start from a site and reach a giant outgoing component follows $P_\infty \sim (q - q_c)^\beta$. For mean-field systems (such as infinite-dimensional systems, random graphs and Cayley trees) it is known that $\beta = 1$ [36]. This regular mean-field result is not always valid. Instead, we study [35] the behavior of (3.46) near $q = q_c$, $u = 1$, and find

$$\beta = \begin{cases} \frac{1}{3-\lambda^\star} & 2 < \lambda^\star < 3, \\ \frac{1}{\lambda^\star - 3} & 3 < \lambda^\star < 4, \\ 1 & \lambda^\star > 4, \end{cases} \tag{3.47}$$

where

$$\lambda^\star = \lambda_{out} + \frac{\lambda_{in} - \lambda_{out}}{\lambda_{in} - 1} \ . \tag{3.48}$$

We see that the order parameter exponent β attains its usual mean-field value only for $\lambda^\star > 4$. As $\lambda_{out} \to \lambda_{in}$ the correlated fraction BA of sites resembles non-directed networks [28, 37] (where there is no distinction between incoming and outgoing degrees). In this case we get $\lambda^\star = \lambda_{out} = \lambda_{in}$ for any amount of correlation A. The criterion for the existence of a giant component is then $\langle k^2 \rangle / \langle k \rangle = 1$, and not 2 as in the non-directed case. The difference stems from the fact that in the non-directed case one of the links is used to reach the site, while in the directed case there is generally no correlation between the location of the incoming and outgoing links. Therefore, one more outgoing link is available for leaving the site.

Without any correlations, $A = 0$, different terms prevail in the analysis and

$$\beta = \begin{cases} \frac{1}{\lambda_{out} - 2} & 2 < \lambda_{out} < 3, \\ 1 & \lambda_{out} > 3. \end{cases} \tag{3.49}$$

This is the same as (3.47) but with $\lambda^\star = \lambda_{out} + 1$.

The GSCC is the intersection of the IN and OUT components. Therefore, it behaves as the smaller of the two components: $\beta_{GSCC} = max(\beta_{in}, \beta_{out})$. This can be also derived by applying the same methods as for the IN and OUT components to the generating function of the GSCC obtained in [32]. The exponent for the GWCC, on the other hand, is independent of the exponents of the other components, since the transition point is different.

Finite Component Sizes

It is known that for a random graph of arbitrary degree distribution the finite clusters follow the scaling form

$$n(s) \sim s^{-\tau} e^{-s/s^*} \ , \tag{3.50}$$

where s is the cluster size and $n(s)$ is the number of clusters of size s. At criticality $s^* \sim |q - q_c|^{-\sigma}$ diverges and the tail of the distribution follows a power law.

The probability that s sites can be reached from a site by following links at criticality follows $p(s) \sim s^{-\tau}$, and is generated by H_0, where $H_0(y) = \sum_s p(s) y^s$. As in [28], $H_0(y)$ can be expanded from (3.42). In the presence of correlations we find [35]

$$\tau = \begin{cases} 1 + \frac{1}{\lambda^\star - 2} & 2 < \lambda^\star < 4, \\ \frac{3}{2} & \lambda^\star > 4. \end{cases} \tag{3.51}$$

The regular mean-field exponents are recovered for $\lambda^\star > 4$. For the uncorrelated case we get [35]

$$\tau = \begin{cases} 1 + \frac{1}{\lambda_{out} - 1} & 2 < \lambda_{out} < 3, \\ \frac{3}{2} & \lambda_{out} > 3. \end{cases} \tag{3.52}$$

Now the regular mean-field results are obtained for $\lambda > 3$.

3.2.4 Summary

In summary, we calculate the percolation properties of directed scale-free networks. We find that the percolation critical exponents in scale-free networks are strongly dependent upon the existence of correlations and upon the degree distribution exponents in the range of $2 < \lambda^\star < 4$. This regime characterizes most naturally occurring networks, such as metabolic networks or the WWW. The regular mean-field behavior of percolation in infinite dimensions is recovered only for $\lambda^\star > 4$.

A connection is found between non-directed and directed scale-free percolation exponents for any finite correlation between the *in*- and *out*-degrees. The correlation between the *in*- and *out*-degrees is responsible for the change in the

Table 3.1. Values of $\lambda^\star$ for the different network components for both correlated and uncorrelated cases

	uncorrelated	*correlated*
$GWCC$	$min(\lambda_{out}, \lambda_{in}) + 1$	$min(\lambda_{out}, \lambda_{in})$
IN	$\lambda_{out} + 1$	$\lambda_{out} + \frac{\lambda_{in} - \lambda_{out}}{\lambda_{in} - 1}$
OUT	$\lambda_{in} + 1$	$\lambda_{in} + \frac{\lambda_{out} - \lambda_{in}}{\lambda_{out} - 1}$
$GSCC$	$min(\lambda_{out}, \lambda_{in}) + 1$	$min(\lambda^*_{out}, \lambda^*_{in})$

critical exponents, and the question whether both incoming and outgoing links lead to the same sites (as in non-directed networks) has no influence on the exponents. In the uncorrelated case, i.e. $P(j,k) = P_{in}(j)P_{out}(k)$, the probability to reach an outgoing component does not bear any dependence upon $P_{in}(j)$. The results are summarized in Table 3.1.

3.3 Spatially Embedded Scale-Free Graphs

The networks studied so far were examples of infinite dimensional networks. They are referred to as infinite dimensional objects since there is no notion of vicinity – every site can connect to every other site with some probability – and since the number of sites in a chemical distance (minimal path length) l from a given site grows exponentially (or faster [18]), which is faster than any power law $N(l) \sim l^d$, expected for a d-dimensional lattice.

Here we describe a method for embedding scale-free networks, with degree distribution $P(k) \sim k^{-\lambda}$, in regular Euclidean lattices accounting for geographical properties [38]. The embedding is driven by a natural constraint of minimization of the total length of the links in the system. All networks with $\lambda > 2$ can be successfully embedded up to an (Euclidean) distance ξ which can be made as large as desired upon the changing of an external parameter. However, the natural cutoff of the distribution can only be achieved for $\lambda > 3$. Clusters of successive layers are found to be compact (the fractal dimension is $d_f = d$), while the dimension of the shortest path between any two sites is smaller than one: $d_{min} = \frac{\lambda - 2}{\lambda - 1 - 1/d}$, contrary to all other known examples of fractals and disordered lattices. An alternative method was suggested by Warren *et al* [39].

All of the networks discussed in previous sections were off-lattice, *i.e.* the Euclidean distance between nodes was irrelevant. However, real-life networks are often embedded in Euclidean geographical space (e.g., the Internet is embedded in the two-dimensional network of routers, neuronal networks are embedded in a three-dimensional brain, etc.). Indeed, in the case of the Internet, indications for the relevance of embedding space is given in [40].

Here we review and extend a method for generating scale-free networks on Euclidean lattices, accounting for geographical properties, and describe some of

its properties [38]. As a guiding principle we impose the natural restriction that the total length of links in the system be minimal.

3.3.1 Model Definition

Our model is defined as follows. To each site of a d-dimensional lattice, of size R, and with periodic boundary conditions, we assign a random connectivity k taken from the scale-free distribution

$$P(k) = Ck^{-\lambda}, \qquad m \le k < K, \tag{3.53}$$

where the normalization constant $C \approx (\lambda - 1)m^{\lambda-1}$ (for K large) [41]. We then select a site at random and connect it to its closest neighbors until its (previously assigned) connectivity k is realized, or until all sites up to a distance

$$r(k) = Ak^{1/d} \tag{3.54}$$

have been explored. (Links to some of the neighboring sites might prove impossible, in case that the connectivity quota of the target site is already filled.) This process is repeated for all sites of the lattice. We show that following this method networks with $\lambda > 2$ can be successfully embedded up to an (Euclidean) distance ξ which can be made as large as desired upon the changing of the external parameter A.

Suppose that one attempts to embed a scale-free network, by the above recipe, in an *infinite* lattice, $R \to \infty$. Sites with a connectivity larger than a certain cutoff $k_c(A)$ cannot be realized, because of saturation of the surrounding sites. Consider the number of links $n(r)$ entering a generic site from a surrounding neighborhood of radius r. Sites at distance r' are linked to the origin with probability $P(k' > (r'/A)^d)$:

$$P\left(k' > \left(\frac{r'}{A}\right)^d\right) = C \int\limits_{(\frac{r'}{A})^d} k^{-\lambda} dk \sim \begin{cases} 1 & r' < A. \\ (\frac{r'}{A})^{d(1-\lambda)} & r' > A. \end{cases} \tag{3.55}$$

Hence

$$n(r) \sim \int\limits_0^r dr' r'^{d-1} P\left(k' > \left(\frac{r'}{A}\right)^d\right) \sim \frac{\lambda - 1}{d(\lambda - 2)} A^d - \frac{A^{d(\lambda-1)}}{d(\lambda - 2)} r^{d(2-\lambda)}. \tag{3.56}$$

The cutoff connectivity is then

$$k_c = \lim_{r\to\infty} n(r) \sim \frac{1}{\lambda - 2} A^d. \tag{3.57}$$

The cutoff connectivity implies a cutoff length

$$\xi = r(k_c) \sim (\lambda - 2)^{-1/d} A^2. \tag{3.58}$$

The embedded network is *scale-free* up to distances $r < \xi$, and repeats itself (statistically) for $r > \xi$, similar to the infinite percolation cluster above criticality: The infinite cluster in percolation is *fractal* up to the coherence length ξ and repeats thereafter [13, 42, 43].

When the lattice is finite, $R < \infty$, the number of sites is finite, $N \sim R^d$, which imposes a maximum connectivity [11, 44]

$$K \sim mN^{1/(\lambda-1)} \sim R^{d/(\lambda-1)}. \tag{3.59}$$

This implies a finite-size cutoff length

$$r_{max} = r(K) \sim AR^{1/(\lambda-1)}. \tag{3.60}$$

The interplay between the three length scales, R, ξ, r_{max}, determines the nature of the network. If the lattice is finite, then the maximal connectivity is $k_{max} = K$ only if $r_{max} < \xi$. Otherwise ($r_{max} > \xi$) the lattice repeats itself at length scales larger than ξ. As long as $\min(r_{max}, \xi) \ll R$, the finite size of the lattice imposes no serious restrictions. Otherwise ($\min(r_{max}, \xi) \geq R$) finite-size effects become important. We emphasize that in all cases the degree distribution (up to the cutoff) is scale-free.

To study the possibility of embedding the network in the lattice we can use (3.57) in conjunction with (3.54). This yields:

$$r_{max} \equiv r(k_c) = (\lambda - 2)^{1/d} k_c^{2/d}\,. \tag{3.61}$$

Since we forbid sites to connect further than the lattice size we must demand $r_{max} \leq R = N^{1/d}$, which means that networks can be embedded in a lattice in the suggested manner only if $k_c \leq N^{1/2}$. This limitation imposes an unnatural cutoff whenever $\lambda < 3$, when compared to (3.14).

In Fig. 3.4a we show typical networks that result from our embedding method, for $\lambda = 2.5$ and 5 in two-dimensional lattices (we limit our numerical results to $d = 2$). The larger λ is the more closely the network resembles the embedding lattice, because longer links are rare [45]. In Fig. 3.4b we show the same networks as in part (a) where successive chemical shells are depicted in different colors. Chemical shell l consists of all sites at minimal distance (minimal number of connecting links) l from a given site. For our choice of parameters, $\lambda = 5$ happens to fall in the region of $\xi > r_{max}$, while for $\lambda = 2.5$, $\xi < r_{max}$. In the latter case we clearly see (Fig. 3.4b, $\lambda = 2.5$) the (statistical) repetition of the network beyond the length scale ξ. The different regimes are summarized in Fig. 3.5.

We now address the geometrical properties of the networks, arising from their embedding in Euclidean space. To this aim, it is useful to consider the spatial arrangement of the networks as measured both in an Euclidean metric and in *chemical space*. The chemical distance l between any two sites is the length of the minimal path between them (*minimal* number of links). Thus if the distance between the two sites is r, then $l \sim r^{d_{min}}$ defines the minimal length exponent

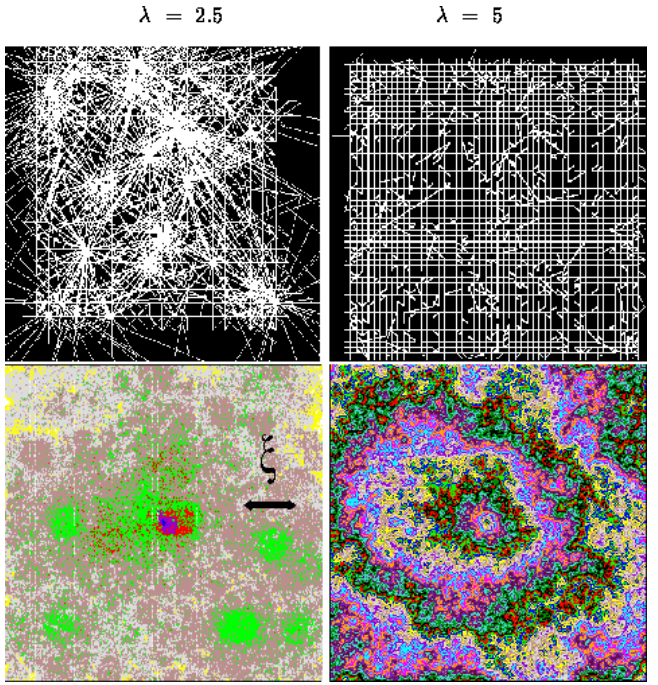

Fig. 3.4. Spatial structure of connectivity network. Top: shown is the typical map of links for a system of 50 x 50 sites generated from a degree distributions with $\lambda = 2.5$ and $\lambda = 5$. Bottom: shown (in different colors) are shells of equidistant sites to the central one in a lattice of 300 x 300 sites. Note that for $\lambda = 5$, shells are concentric and continuous fractals; but for $\lambda = 2.5$, shells are broken

d_{min}. We will see that $d_{min} < 1$ (for $d > 1$), contrary to all naturally occurring fractals and disordered media. Sites at chemical distance l from a given site constitute its l-th chemical shell. The number of (connected) sites within radius r scales as $m(r) \sim r^{d_f}$, defining the fractal dimension d_f. Likewise, the number of (connected) sites within chemical radius l scales as $m(l) \sim l^{d_l}$, which defines the fractal dimension d_l in chemical space. The two fractal dimension are related: $d_{min} = d_f/d_l$ [13, 42, 43].

To study d_f, we compute the perimeter $S(r)$, the number of sites that connect the interior cluster of a region of radius r to sites outside. The fractal dimension then follows from the scaling relation $S(r) \sim r^{d_f - 1}$. We focus on the regime $\xi > r_{max}$. Consider a shell dr', of radius r'. A site of connectivity k' within the shell is connected to the outside (to a distance larger than $r - r'$) with probability $P(k' > (\frac{r-r'}{A})^d)$, (3.55). Thus,

$$S(r) = \int_0^r dr' r'^{d-1} P\left(k' > \left(\frac{r-r'}{A}\right)^d\right) \sim \begin{cases} r^d & r < A, \\ c(\lambda) A r^{d-1} & r > A, \end{cases} \quad (3.62)$$

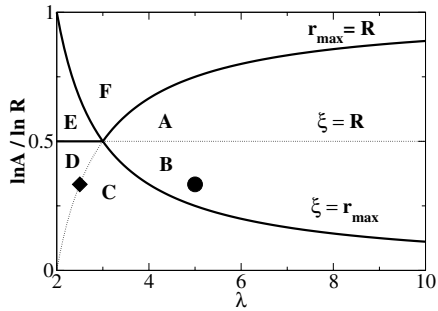

Fig. 3.5. This diagram shows the six regions where different behavior of the network is found: for region A: $r_{max} < R < \xi$, B: $r_{max} < \xi < R$, C: $\xi < r_{max} < R$, D: $\xi < R < r_{max}$, E: $R < \xi < r_{max}$, F: $R < r_{max} < \xi$. The diagram can be mapped into only four regions where the cutoff k_c and where size effect K are expected. A and B: no cutoff and no size effect; C and D: cutoff and no size effect; E: cutoff and size effect; F: no cutoff but size effect. The two symbols indicate the parameters corresponding to Fig. 3.4b, (full diamond) $\lambda = 2.5$ and (full circle) $\lambda = 5$

where $c(\lambda) \sim 1+1/[d(\lambda-1)+1]$. In other words, the network is compact, $d_f = d$ at large distances $r > A$, and super-compact, $d_f = d+1$, at $r < A$.

In order to compute d_{min} (or d_l), we regard the chemical shells as being roughly smooth, at least in the regime $\xi > r_{max}$, as suggested by Fig. 3.4b ($\lambda = 5$). Let the width of shell l be $\Delta r(l)$, then

$$l = \int dl = \int \frac{dr}{\Delta r(l)} \sim r^{d_{min}}, \tag{3.63}$$

since $\Delta l = 1$. The number of sites in shell l, $N(l)$, is, on the one hand, $N(l) \sim r(l)^{d-1}\Delta r(l)$. On the other hand, since the maximal connectivity in shell l is $K(l) \sim N(l)^{1/(\lambda-1)}$, the thickness of shell $(l+1)$ is $\Delta r(l+1)$ which is determined by the length of the largest link to the next shell i.e., $r[K(l)]$, and thus, $\Delta r(l+1) \sim r[K(l)] \sim AK(l)^{1/d}$. Assuming (for large l) that $\Delta r(l+1) \sim \Delta r(l)$, we obtain

$$\Delta r(l) \sim r^{\frac{d-1}{d(\lambda-1)-1}}. \tag{3.64}$$

Using this expression in (3.63), yields

$$d_{min} = \frac{\lambda - 2}{\lambda - 1 - 1/d}. \tag{3.65}$$

Thus, above $d = 1$, the dimensions d_{min} and $d_l = d_f/d_{min}$ are anomalous for all values of λ.

In Fig. 3.6 we plot d_{min} as measured from simulations, and compared with the analytical result (3.65). The scaling suggested in Fig. 3.6b, $N(l) \sim$

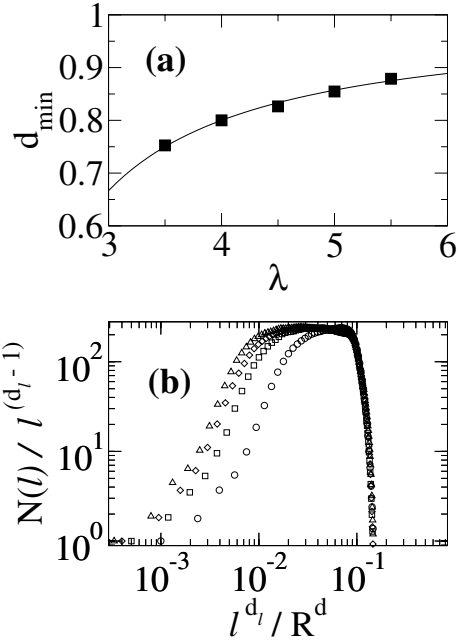

Fig. 3.6. **a** The minimal length exponent d_{min} as a function of λ. Note the good agreement between theoretical estimations (continuous line) and simulations results (full squares). **b** The shape of the $\Phi(l^{d_l}/R^d)$ scaling function is shown for $\lambda = 4$ and several lattice sizes: R=1000 (circle), 2000 (square), 2500 (diamond) and 3000 (triangle)

$l^{d_l-1}\Phi(l^{d_l}/R^d)$, is valid only for $\xi > r_{max}$. For $R \to \infty$, we expect that the network is scale-free up to length scale ξ and the analogous scaling will be $N(l) \sim l^{d_l-1}\Psi(l^{d_l}/\xi^d)$, where $\Psi(x \gg 1) \sim x^{(d-d_l)/d_l}$.

Note on the Upper Cutoff

In (3.14) we suggest that the upper cutoff of a scale free network scales as $N^{1/(\lambda-1)}$. However, for the spatially embedded graphs we find that no graph with $\lambda < 3$ can be embedded in a lattice without sacrificing the natural cutoff (see discussion after (3.61)). That is, the cutoff is limited to $\sqrt{N}$. This holds true for every d. Similar results are indeed obtained for mean field (*i.e.* non-embedded) graphs [46], while Warren *et al* [39] find the natural cutoff even for graphs embedded in $d = 2$ lattices.

A possible explanation is in the different method for the network implementation, which leads to different ensembles. For the non-embedded networks we allowed every link to lead to every other with an equal probability, thus allowing more than one edge between a pair of sites, and edges leading from a site to itself which were just ignored. In contrast, in the spatially embedded case no such connections were allowed. It is plausible that allowing such connections

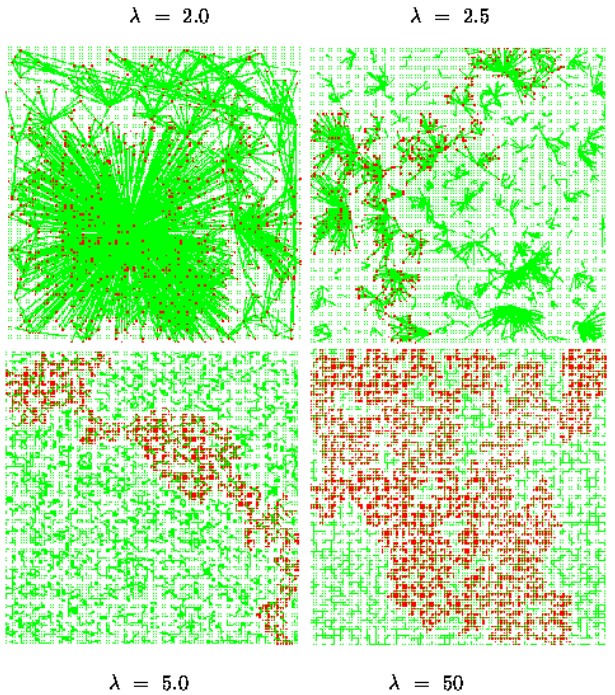

Fig. 3.7. The infinite cluster in scale free networks at criticality. The clusters were generated using a Leath type method, where the nearest available nodes are selected in each shell

or, alternatively, allowing a deviation from the degree distribution, leads to the "natural" cutoff, while requiring the exact degree sequence in conjunction with no such connections influences the ensemble, bringing to an upper cutoff of $\sqrt{N}$, due to the high probability of forming such connections when the cutoff is higher. The limit of $K \sim \sqrt{N}$ seems to stem from the fact that the expected number of edges between two such sites (or self-loops of a single such site) is of order $K^2/\langle k \rangle N \gg 1$, which implies that most networks having such high degree sites will be multigraphs, and therefore this might limit the cutoff. On the other hand, since degree 1 sites consist of a finite fraction of the links in the network, a finite fraction of the links of high degree sites will link to them, implying that the tail of the distribution, and therefore the scaling of the cutoff is not changed, even when double edges and self loops are removed.

3.3.2 Summary

In summary, we propose a method for embedding scale-free networks in Euclidean lattices. The method is based on a natural principle of minimizing the total length of links in the system. This principle enables us to embed the scale-free networks in Euclidean space without additional external exponents. Very

recently, independently, Manna and Sen [47] and Xulvi-Brunet and Sokolov [48] suggested a different embedding method in Euclidean space which include an external exponent. We have shown that while the fractal dimension d_f of the network is the same as the Euclidean dimension, the chemical dimension $d_l > d_f$ for all values of λ, yielding $d_{min} < 1$ for all λ and $d > 1$. A related work by Warren, Sander and Sokolov [39], studies some percolation properties of a similar geographical model. In Fig. 3.7 we show some snapshots of the infinite cluster at the percolation threshold, for $2d$ scale free systems with various values of λ.

Acknowledgments

Daniel ben-Avraham wishes to thank the NSF for support under contract PHY-0140094.

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