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COMMENT

Probability density of the 2D percolation cluster perimeter

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Abstract. We study the form of the probability density $P(\boldsymbol{r}, t)$ of the end-to-end distance r of a segment of length t of the perimeter of the 2D percolation cluster at criticality. We find that $P(\boldsymbol{r}, t)$ behaves as $P(\boldsymbol{r}, t) \propto t^{-2/d_H} \exp[-(r/t^{1/d_H})^{\tilde{\delta}}]$, where $\tilde{\delta} = 1/(1-1/d_H)$. The exponent $d_H = \frac{7}{4}$ is the fractal dimension of the hull (the external perimeter of a percolation cluster). This result fits very well with recent numerical data.

The problem of the perimeter of a percolation cluster has attracted considerable attention in recent years [1-11]. The main reasons for this interest are its closed relation to various physical problems such as diffusion fronts [12], and polymers at the θ point [13]. Scaling theories [5, 10], mapping on the Coulomb gas [11], and numerical simulations [1-4, 6-10, 14] have been applied to find the values of exponents characterising the properties of the percolation perimeter. A surprising result is the strong evidence that only *one* exponent, the correlation exponent ν, is sufficient to characterise the properties of the perimeter. These include properties such as the fractal dimension of the perimeter, d_H, and the distribution n_h of clusters with perimeter length h.

The fractal dimension is defined by

$$t \propto r^{d_H} \tag{1}$$

where r is the end-to-end distance, t the number of steps and d_H was found to be $d_H = 1 + 1/\nu$ [5, 8, 11, 12, 14]. At criticality the distribution n_h has the power-law behaviour [14]

$$n_h \propto h^{-\tau_h} \qquad \tau_h = 1 + 2\nu/(1+\nu). \tag{2}$$

In this comment we study the density distribution $P(\boldsymbol{r}, t)$, which is the probability that t steps along the perimeter have an end-to-end distance $r = |\boldsymbol{r}|$, for a general walk which includes as a particular case the percolation perimeters. We find that also $P(\boldsymbol{r}, t)$ depends only on ν. The form suggested for $P(\boldsymbol{r}, t)$ is found to be in good agreement with recent numerical simulations [10].

The perimeters of a percolation system can be generated as a particular case of the family of dressed self-avoiding walks (DSAW). The general case was defined and studied by Gouyet *et al* [10]. For a square lattice, at criticality, such a walk is completely defined by the probability p_2 $(0 \leq p_2 < 1)$ to step forward. The other probabilities p_1 to turn left and p_3 to turn right are then determined. The mean distance a_{eff} ($a_{\text{eff}} \approx 1/(1-p_2)$) between two turns (right or left for the square lattice case), may be varied from about one step length (more exactly 1.17) when $p_2 = 0$, to infinity when $p_2 = 1$. The percolation cluster perimeters at criticality correspond to $p_2 = p_c(1-p_c) \simeq 0.2414$ (then $p_1 = p_c^2$ and $p_3 = 1 - p_c$).

The distribution function $N(r, t)$ of the DSAW of t steps and end-to-end distance r can be written as the product of the number of surviving (i.e. not yet closed) walks $N_s(t)$ by a reduced scaling distribution $\tilde{N}(r, t) = \tilde{N}(rt^{-1/d_H})$:

$$N(r, t) = t^{-1/d_H} N_s(t) \tilde{N}(rt^{-1/d_H}) \tag{3}$$

or more generally with a scaling function $\mathcal{N}(r/\bar{r}(t))$, where $\bar{r}(t) = (r^2(t, a_{eff}))^{1/2}$ is the average end-to-end distance,

$$N(r, t) = (1/\bar{r}(t)) N_s(t) \mathcal{N}(r/\bar{r}(t)). \tag{4}$$

Expressions (3) and (4) are equivalent when $t > a_{eff}$, since then

$$\bar{r}(t) \equiv (a_{eff})^{1-1/d_H} t^{1/d_H}. \tag{5}$$

The scaling forms (3) and (4) of $N(r, t)$ were suggested by Gouyet *et al* [10] and supported by numerical data.

The probability $P(\boldsymbol{r}, t)$ of obtaining a t steps DSAW with an end-to-end distance $r = |\boldsymbol{r}|$ is related to

$$2\pi r P(\boldsymbol{r}, t) \equiv (1/\bar{r}(t)) \mathcal{N}(r/\bar{r}(t))$$

or (6)

$$P(\boldsymbol{r}, t) = [1/\bar{r}(t)^2] f(r/\bar{r}(t)).$$

In analogy with several other similar problems such as self-avoiding walks (SAW) [15] or anomalous random walks [16] we suggest that $P(\boldsymbol{r}, t)$ is of the form

$$P(\boldsymbol{r}, t) = C_t (r/\bar{r}(t))^{\bar{g}} \exp[-b(r/\bar{r}(t))^{\bar{\delta}}] \tag{7}$$

where from normalisation $\int P(\boldsymbol{r}, t)\, d^2r = 1$, follows $C_t \propto t^{-2/d_H}$. To determine the exponents $\bar{g}$ and $\bar{\delta}$, we use similar arguments to those presented by de Gennes [17] for SAW. The probability of returning to the origin (one step from the origin $|\boldsymbol{r}| = a$) is

$$P(|\boldsymbol{r}| = a, t) \propto C_t (a/\bar{r}(t))^{\bar{g}}. \tag{8}$$

This probability is also equal to the probability of obtaining a cluster of perimeter $t = h$ which is given by

$$h n_h \propto h^{-2/d_H}. \tag{9}$$

From comparison of (8) and (9) it follows that $\bar{g} = 0$.

The exponent $\bar{\delta}$ was found for several similar systems to be [15, 16]:

$$\bar{\delta} = (1 - 1/d_w)^{-1} \tag{10}$$

where d_w characterises how the end-to-end distance of a walk scales with time, $\langle r^2 \rangle \propto t^{2/d_w}$. In our case $d_w = d_H = 1 + 1/\nu$, thus we find that

$$\bar{\delta} = [1 - \nu/(1+\nu)]^{-1} = 1 + \nu = \tfrac{7}{3}. \tag{11}$$

Combining (5), (6) and (11) we expect that

$$\mathcal{N}(r/\bar{r}(t)) = A[r/\bar{r}(t)] \exp[-b(r/\bar{r}(t))^{7/3}]. \tag{12}$$

In order to test the above predictions we compared (12) to the numerical values obtained by Gouyet *et al* (see figure 1). A good fit to the data is obtained using (12).

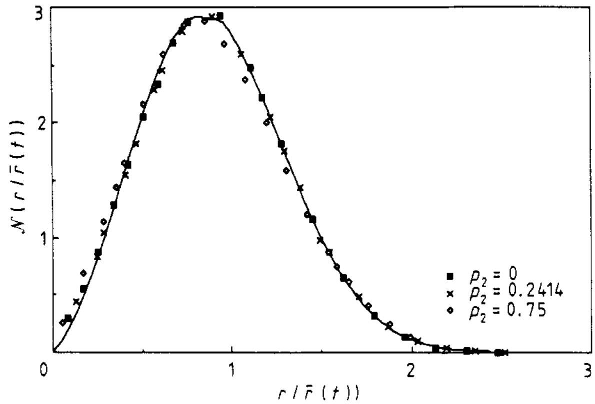

Figure 1. The probability density $\mathcal{N}(r/\bar{r}(t))$ as a function of $r/\bar{r}(t)$ for several DSAW. The different signs represent numerical data taken from [10]. The percolation perimeter corresponds to $p_2 = 0.2414$ (×). The full curve represents the theoretical fit of (13) with $A = 1.792$, $B = 6.612$ and $b = 1.11$. The surface below the curves is not normalised to unity. Note that for small values of $r/\bar{r}(t)$ the numerical data are slightly higher than the theoretical prediction.

However, a much better fit as shown in the figure is obtained using a correction term, i.e.

$$\mathcal{N}(r/\bar{r}(t)) = [A(r/\bar{r}) + B(r/\bar{r})^2] \exp[-b(r/\bar{r})^{7/3}]. \tag{13}$$

For the percolation perimeter, the best fit yield $A = 1.792$, $B = 6.612$ and $b = 1.11$.

In summary it is interesting to point out that the distribution $\mathcal{N}(r/\bar{r}(t))$ seems also to be governed only by a single exponent ν. Also the result $\bar{\delta} = (1 - 1/d_{\mathrm{H}})^{-1}$ supports the apparently rather general conjecture that $\bar{\delta} = (1 - 1/d_{\mathrm{w}})^{-1}$ for a wide family of self-avoiding walks.

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