Range of Validity of the Einstein Relation

In a recent Letter\(^1\) a relation\(^2\) between the fractal dimension \(d_f\), the fracton dimension \(d_s\), and the resistivity exponent \(x\) was considered:

\[
d_f = d_s x / (2 - d_s).
\] (1)

The author concludes from this relation that (i) when \(d_s \to 2\), \(d_f \to \infty\) and the structure is collapsed in all dimensions, and (ii) for \(d_s > 2\) the Einstein relation\(^2\)

\[
d_w = d_f + x
\] (2)

ceases to apply.

However, we argue here that \(d_f\) can be finite when \(d_s \to 2\) and the Einstein relation (Eq. (2)) is valid also for \(d_s \geq 2\). This will be the case if the exponent \(x\) in Eqs. (1) and (2) is interpreted as the exponent characterizing the resistance between the two bars\(^5\) of size \(r^{d_s - 1}\) separated by a distance \(r\) and not as interpreted by Ref. 1 as the resistance between two sites. Both exponents are the same for finitely ramified fractals.\(^6\) However, for infinitely ramified aggregates characterized by \(d_s \gg 2\) the two exponents are not the same in general.\(^6\) This interpretation of \(x\) allows \(x \to 0\) for \(d_s = 2\) and \(d_w = d_f\) to be finite. Also for \(d_s > 2\) it allows \(x < 0\), so that Eq. (2) is still valid.

We present three examples for which \(d_s \geq 2\) where Eqs. (1) and (2) are valid and \(d_f\) is finite. The simplest example is provided by compact clusters of fractal dimension \(d_f = d\). For this case clearly \(d_w = 2\), \(x = 2 - d\), and from \(d_w = 2d_f/d_w\) it follows that \(d_s = d\) and both Eqs. (1) and (2) are valid. Another example is the family of exact fractals studied in Ref. 3. This family includes the case \(d_s = 2\), for which \(d_f\) is found to be finite. Finally, in the following we present a family of random clusters with \(d_s \geq 2\) for which Eqs. (1) and (2) are valid and \(d_f\) is finite when we use our interpretation of \(x\).

The model is a family of random clusters, without loops (trees) and without dead ends, embedded on a Cayley tree with coordination number \(n \geq 3\). In this model we generate trees with adjustable \(d_f \geq 2\). Let \(P(l) = \alpha / l\) be the probability that a site in generation \(l - 1\) will grow to two sites in generation \(l\), and \(1 - P(l)\) be the probability that it will grow to only one site. The expected number of sites that grow from one site in the \((l - 1)\)st generation is \(2 \times P(l) + 1 \times [1 - P(l)] = 1 + P(l)\). Thus the total number of sites \(B(l)\) in the \(l\)th generation is

\[
B(l) = \prod_{l'=1}^{l} [1 + P(l')] = n^l, \quad l \gg 1, \tag{3}
\]

from which the total mass is \(M(l) \sim \sum_{l'=1}^{l} B(l') \sim n^{l+1} \sim r^{l}\). Since trees grown on a Cayley tree have the property that \(r \sim \sqrt{l}\), it follows that \(d_f = 2d_s - 2(\alpha + 1) \geq 2\) (\(\alpha \geq 0\)). The diffusion on these trees was calculated exactly\(^6\) and found to be \(\langle t^2 \rangle \sim t\). Thus one finds \(\langle r^4 \rangle \sim t\) or \(d_w = 4\) and \(d_s = 2d_f/d_w = d_f/2 = d_f\). This result, \(d_w = 4\), can be obtained also from the Einstein relation, Eq. (2). Let \(\rho(l)\) be the resistance between generation \(l\) and all sites in generation \(l\), and let \(\rho_1(l)\) be the resistance between one site in generation \(l\) and a single site in \(l = 1\). Then \(\rho(l) = \rho_1(l)/l^{d_f - 1}\), since \(l^{d_f - 1}\) represents the effective number of parallel paths to the \(l\)th shell. But clearly \(\rho_1(l) \sim l\) (no loops!), so that \(\rho(l) \sim \frac{R^2}{l^{d_f - 1}} \sim R^{3 - d_f} \sim R^{d_f} - R^x\). Substituting \(x = 4 - d_f\) in Eq. (2) we obtain \(d_w = 4\) as found exactly, independently.\(^6\) Notice that if these fractals are embedded in a dimension \(d\), satisfying \(d < d_f\) (\(d_s\) is the upper critical dimension for this model) then

\[
d_w = 2d_f/d_s, \quad d_s = d_f. \tag{4}
\]

These results are generalizations of the results of diffusion on linear chains\(^2\) which are obtained when one substitutes \(d_s = 1\) in Eqs. (4). It should also be noted that Eqs. (4) and the results \(d_w = d_f(1 + 1/d_s)\), \(d_s = 2d_f/(d_f + 1)\) found recently for finitely ramified clusters without loops are special cases of a general relation given by Havlin et al.\(^7\)

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