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LETTER TO THE EDITOR

Non-linear response in percolation systems

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Abstract. We study diffusion in $d = 2$ percolation clusters at criticality under the influence of an external time-dependent field $E(t) = E_0 \sin(\omega t)$. We find that the mean displacement $\langle x(t)\rangle$ of a random walker is a periodic function with a phase shift $\varphi \simeq \frac{1}{3}\pi$ independent of ω and E_0. The amplitude $A(E_0, \omega)$ of $\langle x(t)\rangle$ is linear in E_0 below a crossover field $E_0^*(\omega)$. Above $E_0^*(\omega)$, $A(E_0, \omega)$ is strongly non-linear, showing a maximum as a function of E_0. We find that *both* the linear and the non-linear regime can be described by a *single* scaling function: $A(E_0, \omega) \sim E_0 \omega^{-2/d_w} F(E_0/E_0^*(\omega))$ where $d_w \simeq 2.87$ is the diffusion exponent and $E_0^*(\omega) \sim \omega^\beta$ with $\beta \simeq 0.3$. The linear response regime is reached for $E_0/\omega^\beta \ll 1$.

In recent years, the problem of transport in random media has been studied extensively. While some progress has been achieved in understanding unbiased diffusion (see, e.g., [1–5]) or diffusion under the influence of a constant bias field [6–11], studies of transport in the presence of an external time-dependent field have been mainly restricted to the linear response regime [12–14].

In this letter we study how the motion of a random walker in a disordered structure is affected by a time-dependent alternating field with *arbitrary* field strength. For specificity, we consider the incipient infinite percolation cluster [5] in $d = 2$ and apply a sinusoidal field

$$E(t) = E_0 \sin(\omega t). \qquad (1)$$

We find that under the influence of this field the mean displacement $\langle x(t)\rangle$ of the walker is a periodic function with a phase shift of $\varphi \simeq \frac{1}{3}\pi$ relative to the external field. While φ is independent of ω and E_0, the amplitude $A(E_0, \omega)$ is non-linear in E_0: for small ω, $A(E_0, \omega)$ exhibits a maximum as a function of E_0. This non-linear behaviour results from the interplay of two competing processes: (i) the field induces a drift which tends to increase the amplitude with increasing field strength E_0 and (ii) the field drives the walker into the dangling ends, an effect by which the amplitude is decreased with increasing E_0. For a *constant* bias field, the second process is dominant [11] and the motion of the walker is ultra-anomalously slow, characterised by a logarithmic time dependence [9, 11]. In contrast, for the *time-dependent* field (1), both processes are relevant which gives rise to the maximum in $A(E_0, \omega)$. We develop a scaling theory for $A(E_0, \omega)$ and find that a *single* scaling function adequately describes the amplitude in *both* the linear and the non-linear E_0 regime.

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In uniform non-random structures the mean displacement $\langle x(t)\rangle$ of a random walker under the influence of the field (1) is given by

$$\langle x(t)\rangle = A(E_0, \omega)[\sin(\omega t-\varphi)+\Delta] \tag{2}$$

where the phase shift is $\varphi=\frac{1}{2}\pi$, the amplitude $A(E_0,\omega)$ is proportional to E_0/ω and $\Delta=1$. Since the current $j(t)$ induced by the field is proportional to $\mathrm{d}\langle x(t)\rangle/\mathrm{d}t$, (2) implies that the real part of the frequency-dependent conductivity $\sigma(\omega)$ is independent of ω.

In random structures, for sufficiently small fields where linear response theory is valid, we expect that the form of (2) is maintained, but the amplitude, phase shift and Δ are changed. To see this, consider $\sigma(\omega)$ which is related to the mean square displacement $\langle r^2(t)\rangle$ of a random walker (without field) by [12]

$$\sigma(\omega)\sim -\omega^2 \lim_{\eta\to 0}\int_0^\infty \exp(-\mathrm{i}\omega t)\exp(-\eta t)\langle r^2(t)\rangle\,\mathrm{d}t. \tag{3}$$

On self-similar structures (fractals) such as the incipient infinite percolation cluster, diffusion is anomalous [1] and

$$\langle r^2(t)\rangle \sim t^{2/d_\mathrm{w}} \tag{4}$$

where d_w is the diffusion exponent, $d_\mathrm{w}\simeq 2.87$ [15] in $d=2$ percolation. From (3) and (4) we obtain

$$\sigma(\omega)\sim \exp[\mathrm{i}(1-2/d_\mathrm{w})\pi/2]\omega^{1-2/d_\mathrm{w}}. \tag{5}$$

Since the Fourier transformed current $j(\omega')$ is proportional to $\sigma(\omega')E(\omega')$ and $E(\omega')=E_0\delta(\omega-\omega')$, it follows from (2) and (5) that in the linear response regime, phase shift and amplitude are given by

$$\varphi=\pi/d_\mathrm{w} \tag{6a}$$

and

$$A(E_0,\omega)\sim E_0\omega^{-2/d_\mathrm{w}}. \tag{6b}$$

These predictions are based on the assumption that the form of (2) remains unchanged for random structures in the linear response regime. Since for constant bias fields non-linear effects are strongly pronounced we expect that larger values of E_0 give rise to non-linear behaviour and to strong deviations from (2) and (6) also for time-dependent fields.

In order to study the linear and non-linear regime we have performed numerical simulations using the exact enumeration method (see, e.g., [15]). This numerical method enables us to calculate exactly the distribution function $P(\boldsymbol{r},t)$ of a random walker on a given percolation cluster for a fixed starting point $\boldsymbol{r}_0$. First we generate a percolation cluster on a square lattice at criticality, using the Leath algorithm [5]. We assume a time-dependent bias field, (1), along the xy direction and choose the transition rates $W_{\boldsymbol{r},\boldsymbol{r}+\delta}$ between neighbouring cluster sites $\boldsymbol{r}$ and $\boldsymbol{r}+\delta$ accordingly†:

$$W_{\boldsymbol{r},\boldsymbol{r}+\delta}=\begin{cases}\frac{1}{4}(1+E(t)) & \delta=(1,0)\text{ or }(0,1)\\ \frac{1}{4}(1-E(t)) & \delta=(-1,0)\text{ or }(0,-1)\end{cases} \tag{7}$$

where by definition $|E(t)|\leqslant 1$.

† By this choice, (7), we simulate the 'blind' ant. If a jump is not allowed because the attempted site does not belong to the cluster then the walker stays in place.

Then the time evolution of $P(\boldsymbol{r}, t)$ is calculated as follows. At $t=0$, the walker starts at the origin, i.e. $P(\boldsymbol{r}, 0)=\delta_{\boldsymbol{r},0}$. At $t=1$, the walker steps with probability $W_{0,0+\delta}$ to the neighbouring cluster sites δ. Hence

$$P(\boldsymbol{r}, 1)=\begin{cases} W_{0,\delta} & \text{for } \boldsymbol{r}=\delta \\ 1-\sum_{\delta} W_{0,\delta} & \text{for } \boldsymbol{r}=0. \end{cases} \tag{8}$$

For $\boldsymbol{r}\neq 0$ and $\boldsymbol{r}\neq\delta$, $P(\boldsymbol{r}, 1)\equiv 0$. By iterating this procedure we find $P(\boldsymbol{r}, 2)$, etc. From $P(\boldsymbol{r}, t)$ we obtain the mean displacement

$$\langle \boldsymbol{r}(t)\rangle \equiv (\langle x(t)\rangle, \langle y(t)\rangle)=\sum_{\boldsymbol{r}} \boldsymbol{r}P(\boldsymbol{r}, t) \tag{9}$$

of the random walker with starting point at the origin for the considered cluster. In order to obtain the corresponding configurational averaged quantities one has to average over many clusters. For our actual computations, we generated clusters up to 150 shells and averages over 200 configurations have been made.

Representative results of $\langle x(t)\rangle$ for two values of E_0 and constant frequency are shown in figure 1. For small fields (a) $\langle x(t)\rangle$ is accurately described by a sinusoidal curve, (2). In contrast, for large fields (b) strong deviations from sinusoidal behaviour occur. Nevertheless, the curve is still periodic and can be characterised by amplitude $A(E_0, \omega)$ and a phase shift φ.

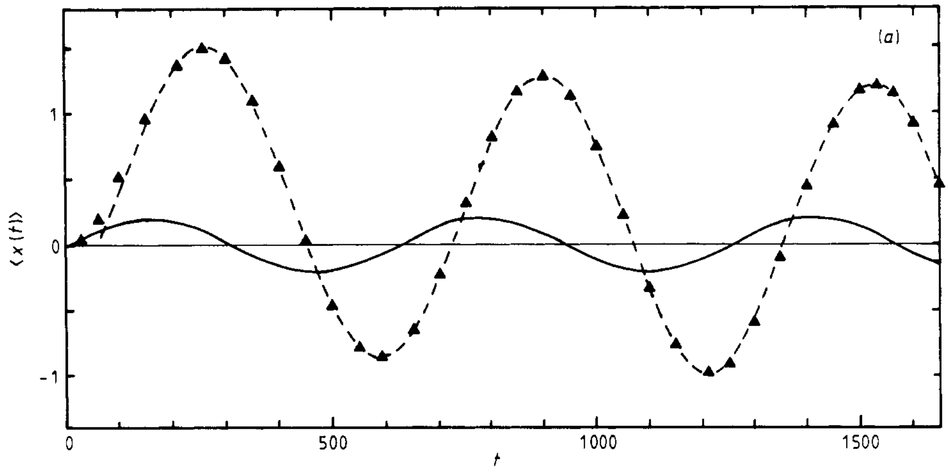
Figure 2 shows our results for $A(E_0, \omega)$ and φ in the whole range of E_0 up to its maximum value $E_0=1$ and various values of ω. While for all considered values of E_0 and ω the phase shift φ is practically constant and accepts the linear response value from (6a), the amplitude for large fields behaves differently from the amplitude for small fields. For sufficiently small fields and large ω our results for $A(E_0, \omega)$ (figure 2) are described by linear response theory, (6b). For large fields and small frequencies strong deviations from linear response theory occur. To see this, we have plotted in figure 3(a) $A(E_0, \omega)$ as a function of E_0 for various values of ω. Below a certain crossover field $E_0^*(\omega)$, A is linear in E_0 representing the linear response regime. Above $E_0^*(\omega)$, strong non-linear effects are observed. The amplitude $A(E_0, \omega)$ exhibits a maximum at $E_0=E_0^*(\omega)$. The crossover field decreases with decreasing ω. The occurrence of this maximum is a consequence of two competing processes which are caused by the bias field (1). On the one hand, the field induces a drift on the walker. If the field is increased the drift and correspondingly the amplitude of the walk tends to increase. On the other hand, the field drives the walker into dangling ends. This effect tends to decrease the amplitude and becomes more pronounced when the field is increased. For small fields and large frequencies the drift is dominant and $A(E_0, \omega)$ increases linearly with E_0, while for large fields and small frequencies the walker can easily get stuck in the dangling ends; the second process is dominant and $A(E_0, \omega)$ decreases with increasing E_0.

In the linear response regime, the amplitude is given by (6b). In order to generalise this result to the non-linear regime we assume that $A(E_0, \omega)$ can be written as

$$A(E_0, \omega)=E_0\omega^{-2/d_w}F(E_0/\omega^{\beta}) \tag{10}$$

where $F(x)$ is a scaling function. In order to satisfy (6b) we must have $F(x)=\text{constant}$ for $x\ll 1$. By (10), the crossover field $E_0^*(\omega)$ is identified by ω^{β}.

For testing the scaling assumption (10) and for determining the crossover exponent β and the scaling function we have plotted $A(E_0, \omega)/(E_0\omega^{-2/d_w})$ as a function of E_0/ω^{β}



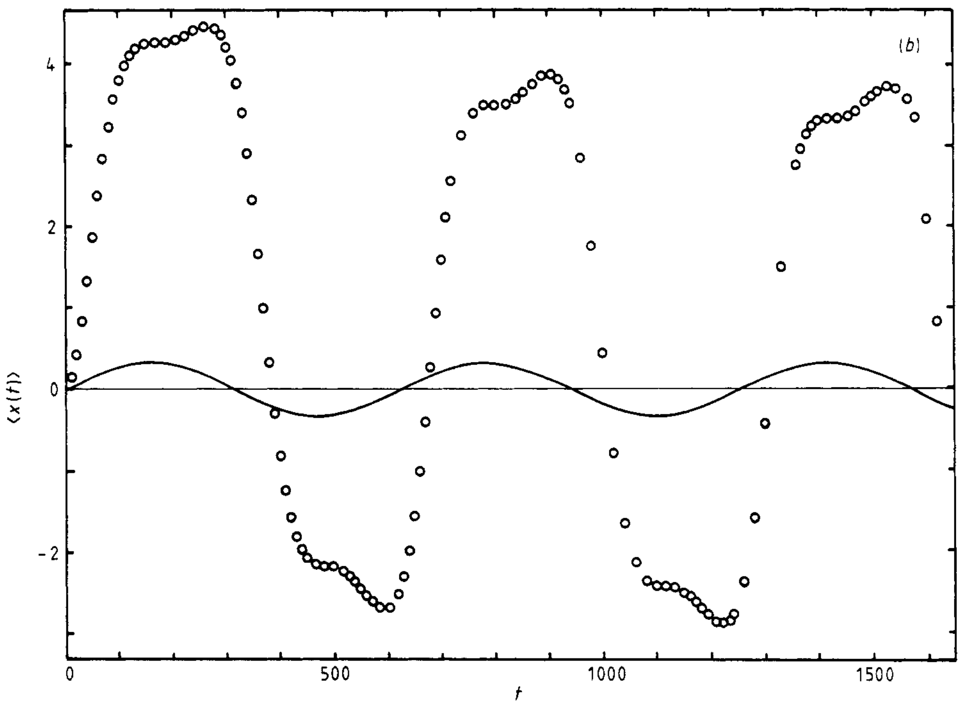


Figure 1. Mean displacement $\langle x(t)\rangle$ of a random walker on the incipient percolation cluster under the influence of an external time-dependent field along the xy direction $\boldsymbol{E}(t)=(E_0, E_0)\sin(\omega t)$ for $\omega=0.01$. (a) shows $\langle x(t)\rangle$ for $E_0=0.1$. The broken curve represents a sinusoidal fit. (b) shows $\langle x(t)\rangle$ for the maximum field strength, $E_0=1$. Strong deviations from sinusoidal behaviour occur. For comparison, the time dependence of the field is also shown (continuous curve). In both cases, the phase shift between $\langle x(t)\rangle$ and $\boldsymbol{E}(t)$ is roughly the same. In contrast to uniform systems, (2), $\langle x(t)\rangle$ becomes negative and thus Δ is considerably smaller than one.

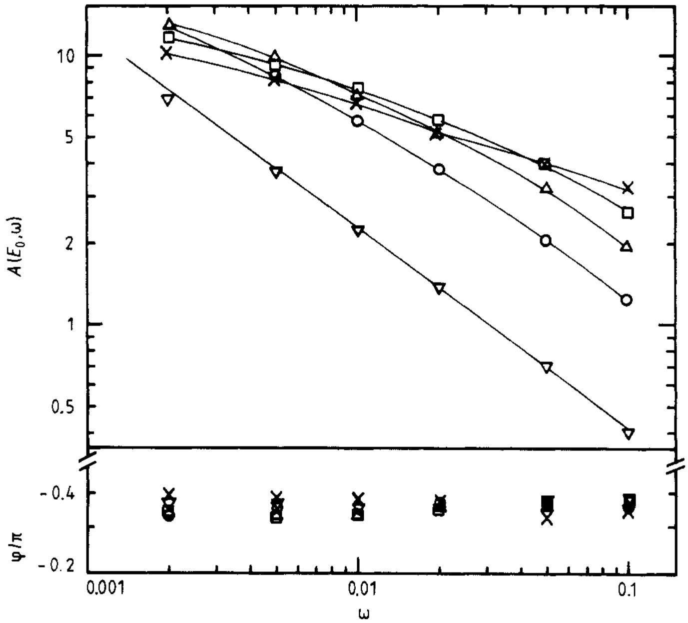


Figure 2. Amplitude $A(E_0, \omega)$ and phase shift φ as a function of ω for various values of field E_0: 0.1 (∇), 0.3 ($\bigcirc$), 0.5 ($\triangle$), 0.7 ($\square$), 1 ($\times$). The amplitude was obtained from the difference between maxima and successive minima of $\langle x(t)\rangle$ (see figure 1).

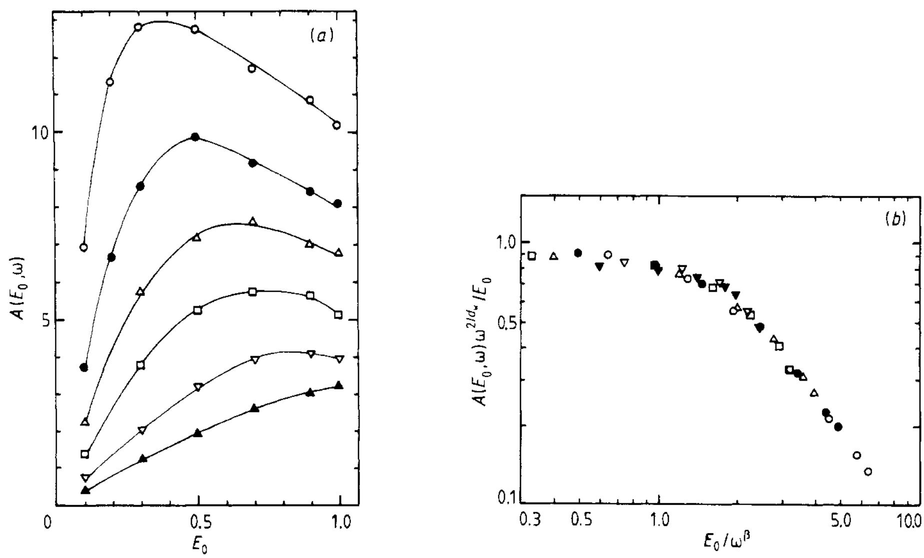


Figure 3. (a) Plot of $A(E_0, \omega)$ as a function of E_0 for various values of frequency ω: 0.002 ($\bigcirc$), 0.005 ($\bullet$), 0.01 ($\triangle$), 0.02 ($\square$), 0.05 (∇), 0.1 ($\blacktriangle$). (b) Plot of $A(E_0, \omega)\omega^{2/d_w}/E_0$ against E_0/ω^{β} for $\beta = 0.3$. The data collapse supports the scaling assumption (10).

and have varied β. We have found that the best data collapse occurred at $\beta \simeq 0.3$; the result is shown in figure 3(b). At the moment we do not have an explanation for this value. For large x, the scaling function shows power law behaviour, $F(x) \sim x^{-\alpha}$, where $\alpha \simeq 1.5$. The data collapse in figure 3(b) confirms our scaling assumption and shows that both the linear and the non-linear regime can be described by a single scaling function.

We also calculated the fluctuation of the mean displacement, $\langle(\Delta \boldsymbol{r})^2\rangle \equiv \langle r^2(t)\rangle - \langle \boldsymbol{r}(t)\rangle^2$, as a function of time for fixed ω and various values of E_0. The data showed clearly that $\langle(\Delta r)^2\rangle \sim t^{2/d_w}$, $d_w = 2.87$, following the same asymptotic law as for the non-biased case. This is expected since the fluctuations are governed by pure diffusion.

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