Trapping problem on a line with a dichotomous disorder

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We study the problem of random trapping on a linear chain when a random walker moves under the influence of a dichotomously disordered field to a neighboring site. The transition probability for moving to the right at each site is chosen with equal probability to be $\frac{1}{2}(1+E)$ or $\frac{1}{2}(1-E)$. We find that the long-time survival probability has the form $S(t) \sim A(c,E)t^{-b(c,E)}$ where $b(c,E)=2\ln[1/(1-c)]/\ln[(1+E)/(1-E)]$, c is the concentration of the traps, and A is a constant. For short times our theory suggests that the survival distribution is log-normally distributed, i.e., $S(t) \sim \exp[-d(\ln t)^2]$. These results are supported by numerical simulations.

A problem of considerable present interest in the physical sciences is that of transport in disordered media.[1,2] The results find application in a wide variety of fields exemplified by solid-state physics,[3] reaction kinetics,[4] and chromatographic processes.[5] Most analysis of transport in a random medium make the assumption that the medium is infinite and calculate quantities like the mean-square displacement, $\langle r^2(t) \rangle$, as a function of time. It is generally believed that if one understands transport in an infinite medium, the results can be used to deduce the dependence of properties like first passage times to absorbing boundaries in finite media on the underlying parameters. In this paper we present results of a study of the classical trapping problem[6] in one dimension for random walks on a line in which a given site of the underlying lattice has an associated transition probability for moving to the right equal to $\frac{1}{2}(1+E)$ or $\frac{1}{2}(1-E)$, each occurring with probability $\frac{1}{2}$, where $E<1$. Some properties of random walks such as the mean-square displacement on such disordered lattices have been given by Sinai,[7] and a more complete analysis is found in a paper by Kesten.[8] It is shown, in the present work, that the asymptotic form of the survival probability is not readily predicted on the basis of transport properties in infinite medium. Rather, our results suggest that the asymptotic form of the survival probability is dominated by a particular configuration of transition probabilities, as well as the large trap-free regions. This particular configuration, which is dominant in the calculation of the survival probability, does not affect the mean-square displacement as found by Sinai. The analysis of this model suggests a novel power-law time dependence for the asympotic survival probability and leads to a phase transition with respect to the behavior of the mean first-passage time.

The trapping problem on a translationally invariant lattice has been studied by a large number of investigators whose main focus of interest has been the survival probability $S(t)$ of a diffusing particle (or random walker) at time t. One asymptotic result is known rigorously for diffusion in translationally invariant spaces. This is due to Donsker and Varadhan,[9] and states that for t sufficiently large

$$-\ln S(t) \sim \{\ln[1/(1-c)]\}^{2/(2+D)} t^{D/(D+2)} , \tag{1}$$

where D is the spatial dimension and c is the concentration of randomly distributed, uncorrelated traps. A similar result has been found by scaling arguments and simulations for trapping on fractals[10,11] with the ordinary dimension D replaced by the fraction dimension,[12] d_s. We consider the problem of finding a similar asymptotic approximation for a random walk in one dimension when the transition probabilities are dichotomous random variables whose values are chosen as described earlier. That method of choosing the transition probabilities ensures that the random walk will be symmetric on the average.

It is straightforward to study the survival probability in the trapping problem in one dimension since the presence of (perfect) traps divides the line disjoint segments. An analytic argument somewhat similar to that given by Grassberger and Procaccia[13] permits us to suggest an asymptotic form for $S(t)$. Let $S(L,t)$ be the survival probability of a random walker whose initial position is uniformly distributed on an interval consisting of L points. There are 2^L possible configurations of transition probabilities, each of which, in general, leads to a different survival probability. We follow Grassberger and Procaccia in assuming that the asymptotic form of the overall survival probability will mainly be determined by random walks on the largest intervals. A second approximation consists in the assumption that the survival probability for a given L can be determined at long times by a single dominant configuration of transition probabilities. This configuration, shown in Fig. 1, leads to the longest survivals among all possible configurations by tending to concentrate the random walkers into the central part of the segment rather than at the ends of the interval in the vicinity of the traps.

Our strategy in estimating the form of $S(L,t)$ for this

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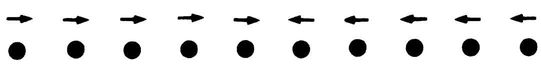

FIG. 1. The configuration with oriented fields which leads to the largest survival among all possible configurations. The arrows indicate the preferred direction for particle movement. This direction is characterized by a probability $P=(1+E)/2$.

particular configuration is to calculate the mean first-passage time to either absorbing point, t^*, and appeal to the result of Newell[14] to infer that $S(L,t)$, for large enough L and t, is expressed

$$S(L,t)\sim\exp(-t/t^*)\ . \tag{2}$$

A standard calculation[15] of t^* shows that the leading-order term in the expression for this quantity, neglecting some unimportant constants, is

$$t^*=T\exp\{(L/2)\ln[(1+E)/(1-E)]\}\ , \tag{3}$$

where T is a constant of order 1 with the dimensions of time. The survival probability is obtained by averaging over all values of L. Since only large L is of interest we may replace the probability distribution for L by a probability density $\psi(L)$ given by

$$\psi(L)=\lambda^2 L\exp(-\lambda L)\ , \tag{4}$$

where $\lambda=\ln[1/(1-c)]$. Thus, the asymptotic form of the survival probability can be calculated from the integral

$$S(t)\simeq\lambda^2\int_0^\infty L\exp\{-(t/T)\exp[-b(E)L] -\lambda(c)L\}dL\ , \tag{5}$$

where

$$b(E)=(\tfrac{1}{2})\ln[(1+E)/(1-E)]\ .$$

The resulting integral can be evaluated approximately by using Laplace's method, leading, finally, to

$$S(t)\simeq K[\ln(t/T)+\ln(b/\lambda)]t^{-b/\lambda}\ , \tag{6}$$

where K is a constant.

The interesting term in this last equation is $t^{-b/\lambda}$. When $b/\lambda>1$ the mean first-passage time will be finite, but when $b/\lambda<1$ it will be infinite. Hence there will be a phase transition when the bias parameter E is sufficiently large. The critical value of this parameter is

$$E_c=[1-(1-c)^2]/[1+(1-c)^2] \tag{7}$$

so that when $c=0$, $E_c=0$, and when $c=1$, $E_c=1$. When $E>E_c$ the mean first-passage time will be infinite, otherwise it will be finite.

We studied this problem numerically using two methods. The first is based on calculating the following exact expression for the survival probability:

$$S(t)=c^2\sum_{L'=1}^{\infty}(1-c)^{L'}(1/2^{L'})\sum_{i=1}^{2^{L'}}S_i(L',t)\ , \tag{8}$$

in which $S_i(L,t)$ is the survival probability for configuration i. For large concentrations, $c\to 1$, and for large t, the sum in Eq. (8) converges rapidly. We have used this equation by exactly enumerating the survival probabilities $S_i(L,t)$ for all configurations of size L up to $L=20$. We chose values of the trap concentration c and the step number t to ensure that Eq. (8) has converged with a reasonable amount of computer time. The results for $S(t)$ for several concentrations and fields are plotted in Fig. 2. The lines represent the theoretical predictions of Eq. (6) and are in good agreement with the data. The second numerical method for calculating survival probabilities is based on generating random configurations of traps and fields and using the exact enumeration method described in Ref. 16. The results for large concentrations coincide with the results found by using our first method. Equation (6) describes the data quite well as may be seen from Fig. 2. For small concentrations or at short times we do not expect Eq. (6) to lead to a good approximation to the survival probability and indeed large deviations are

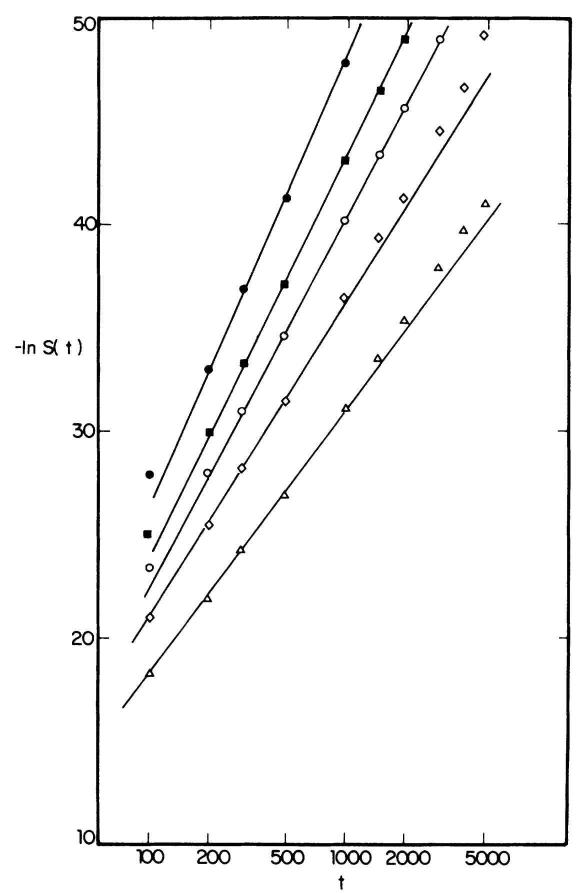

FIG. 2. Plot of $-\ln S(t)$ as a function of $\ln t$ for $E=0.5$ and large trap concentrations: $c=0.995$ (●), 0.99 (■), 0.985 (○), and 0.95 (△). The data were obtained using the first numerical method described by Eq. (8). The straight lines support the theoretical prediction of Eq. (6). The solid line represents the theoretical slopes b/λ in Eq. (6). The slight deviations from the theory at large t are attributed to the finite size of our lattice.

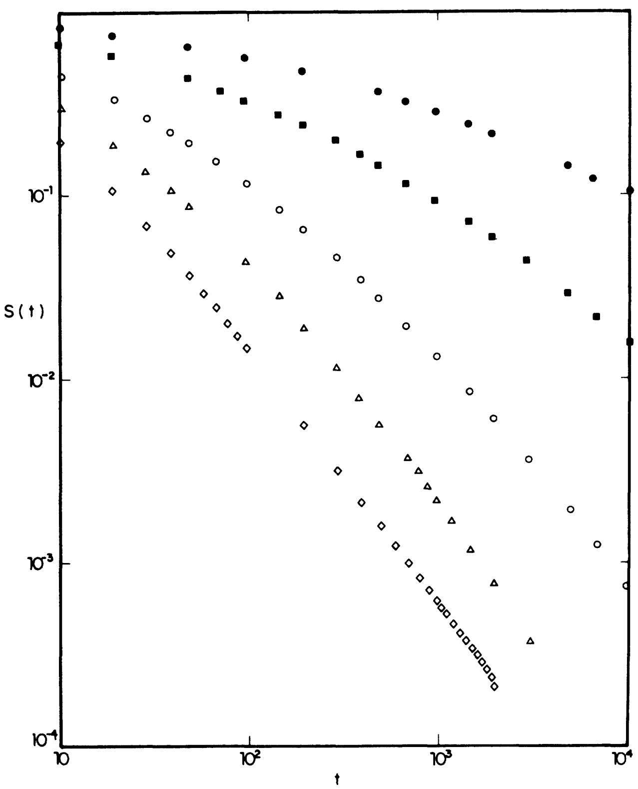

FIG. 3. Plot of $\ln S(t)$ as a function of ln for $E=0.5$ and small concentration of traps: $c=0.05$ (●), 0.1 (■), 0.2 (●), 0.3 (△), and 0.4 (◇). The curves indicate deviations from the predictions of Eq. (6) at short times.

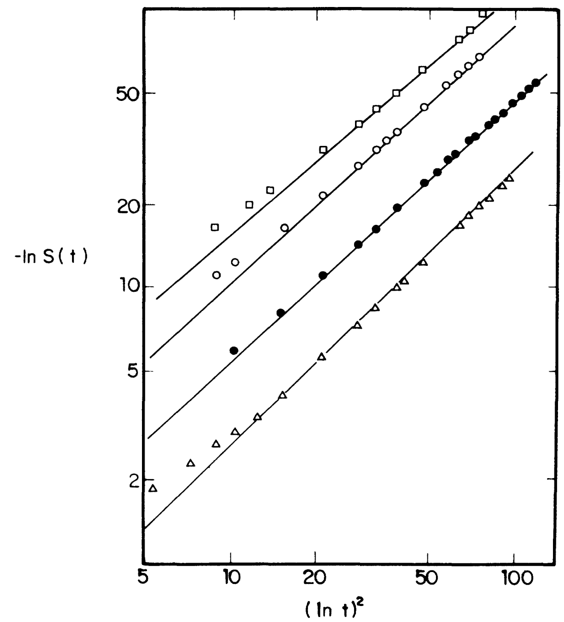

FIG. 4. Plot of $S(t)$ as a function of $(\ln t)^2$ for the data shown in Fig. 3 on a double logarithmic graph. The slopes are 1 ± 0.02, in very good agreement with Eq. (11).

found from the prediction of Eq. (6). These deviations are evident from the graphs in Fig. 3. An explanation for these deviations is that in this range the dominant contribution to $S(L,t)$ does not come from the walkers concentrated in the centers of the line segment as is described by the exponential survival probability shown in Eqs. (2) and (3). At short times we use a different argument to suggest an analytic form for the survival probability. This approach is based on a result of Sinai[7] that the mean-square displacement scales as

$$\langle x^2 \rangle \sim (\ln t)^4 \ . \tag{9}$$

We have accordingly conjectured the following form of the survival probability for short times to hold in a trap-free region of length L,

$$S(L,t) \sim \exp[-\exp(-L^{1/2})t] \ . \tag{10}$$

This form for the component survival probability was indeed supported by the numerical simulations. On evaluating the integral in Eq. (5) with this form of $S(L,t)$ in the integrand we find for $S(t)$:

$$S(t) \sim \exp[-d(\ln t)^2] \ , \tag{11}$$

where the constant d depends both on the trap concentration c and the bias parameter E. Numerical data supporting Eq. (11) are shown in Fig. 4.

An interesting feature of our analysis is the existence of a new characteristic time t^*, represented by Eq. (3), which appears to dominate the long-time behavior of the survival probability. This characteristic time differs from the time calculated using Sinai's result [Eq. (9)]. Only the short-time behavior of $S(t)$ is suggested by the mean-square displacement property given in Eq. (9). Another interesting feature of our analysis is the existence of a "phase transition" in the average survival time from a finite to an infinite mean time to trapping in the random system, depending on the critical bias as given in Eq. (7).

ACKNOWLEDGMENTS

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