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LETTER TO THE EDITOR

Multifractal nature of diffusion on hierarchical structures

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Abstract. We study the moments of the first passage time t of random walks on hierarchical structures using the exact enumeration method. We find that the different moments scale with the size of the system with an infinite hierarchy of exponents characterising a multifractal behaviour.

The study of physical systems with multifractal features has been very active recently. Examples include complicated systems such as the growth probabilities of DLA structures and the voltage distribution of random resistor networks. For recent reviews on multifractals see Paladin and Vulpiani [1] and Stanley and Meakin [2]. The multifractal nature is represented by the fact that different moments of a physical property are characterised by an infinite set of independent exponents and not by a simple power of the first moment.

In this letter we present a simple model, a one-dimensional hierarchical structure, on which the diffusion has multifractal behaviour. The hierarchical structure [3] studied here is shown in figure 1. The powers R^m represent the teeth length in the structure. The random walker can step on the backbone and on the teeth of the comb, each step being of one unit length. We study the moments of the first passage time to pass the nth generation of the hierarchical structure as a function of the size $L=2^n$ of the system. This is done by using the exact enumeration method [4] for the random-walk probability density under the boundary condition that the two edges of the nth generation are absorbing sites. In this case the sum over all sites of the probability density in a given time t is the survival probability $S_L(t)$ at this time. Results for $S_L(t)$ for different values of L were calculated. It is expected, due to the finite size of the systems, that for long enough time the survival will decay exponentially as

$$S_L(t) = A\exp(-bt/t^*) \qquad t \gg 1. \tag{1}$$

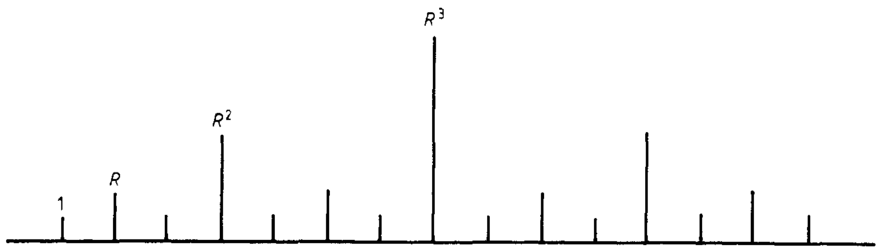

Figure 1. Hierarchical comb structure. The random walker makes unit steps on the backbone and on the teeth of the structure. The powers R^n represent the teeth length in the structure. The third generation of the hierarchical structure is shown here.

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Indeed, the results found for $S_L(t)$ for long times are as predicted by (1). Results for $t^*(L)$ are given in figure 2.

The average t (the first moment of the first passage time) is given by $\langle t\rangle = \Sigma_t\, S_L(t)$, and the results are plotted as a function of L in figure 2. It is seen that the slope representing t^* is 3.06 ± 0.06 (for $R = 3$) and is different from the slope representing $\langle t\rangle$, 2.58 ± 0.02. This shows the surprising result that different characteristic times (t^* and $\langle t\rangle$) scale differently with the size of the system, namely

$$t^* \sim L^{3.06} \qquad \langle t\rangle \sim L^{2.58}. \tag{2}$$

The above feature is a first indication of the multifractal nature of the diffusion on the hierarchical structures. A qualitative explanation for this behaviour is as follows. Whereas the first moment is characterised by relatively short times, the exponential behaviour, equation (1), is dominated by the very long characteristic times, characterising large moments, which scale differently from the short times.

The different moments can be calculated from $S_L(t)$ by

$$\langle t^q\rangle = -\sum_t t^q\, \mathrm{d}S_L(t)/\mathrm{d}t. \tag{3}$$

This is so since the derivative $-\mathrm{d}S/\mathrm{d}t$ in (3) represents the probability to be absorbed by the boundaries at time t. Using the exact numerical values obtained for $S_L(t)$ we find the values for the different moments. These results show that the moments scale as

$$\langle t^q\rangle \sim L^{\tau(q)}. \tag{4}$$

The values of $\tau(q)$ are plotted in figure 3. This figure shows a curvature in the small-q regime (note the deviation from the broken line through $\tau(4)$ and $\tau(5)$) which is characteristic of multifractals.

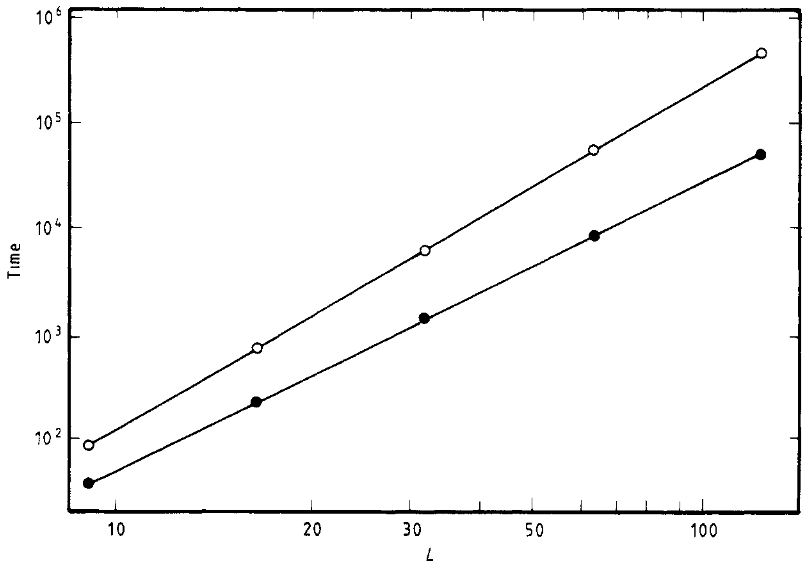

Figure 2. Plot of two characteristic times (t^*(○) and $\langle t\rangle$(●)) as a function of the size L of the system on a log-log plot for $R = 3$. The different slopes represent the different exponents in (2).

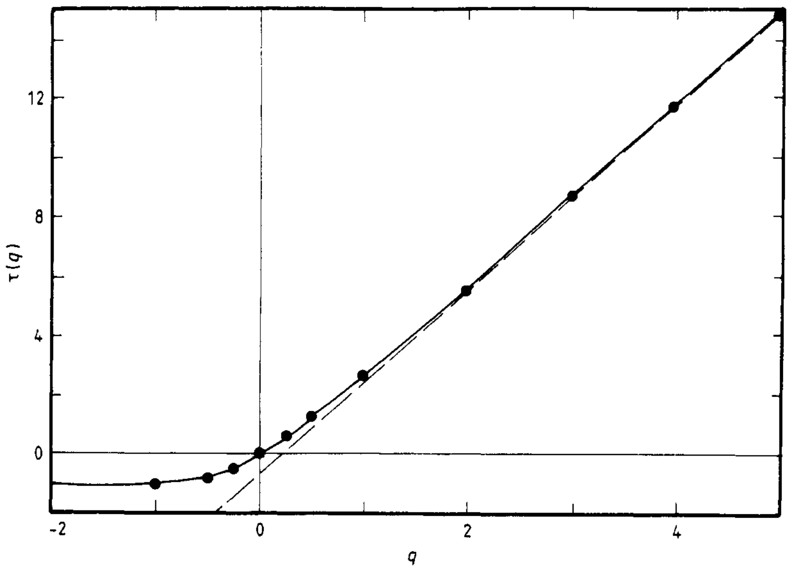

Figure 3. Plot of $\tau(q)$ as a function of q. The broken line represents the straight line between $\tau(4) = 11.70 \pm 0.04$ and $\tau(5) = 14.80 \pm 0.04$. The slope of the broken line is different from the slope of $\tau(q)$ for small q showing the multifractal feature of the problem.

The exact enumeration method used here enables us to solve the diffusion problem numerically with very small error bars determined by the limits of the computer accuracy (10^{-16}) and thus to observe the multifractal nature of the first passage time problem. This result is not consistent with the scaling hypothesis used recently in [5] which led to a *gap* exponent for the first passage time moments (see also equations (77) and (78) in [6]). A more detailed study including the moments of the mean-square displacement on hierarchical structures and explaining some of the exponents found in the present work will be published elsewhere.

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