Cluster growth model for treelike structures

S. Havlin* and R. Nossal

Physical Sciences Laboratory, Division of Computer Research Technology,
National Institutes of Health, Bethesda, Maryland 20205

B. Trus

Computer Systems Laboratory, Division of Computer Research Technology,
National Institutes of Health, Bethesda, Maryland 20205

(Received 10 September 1984; revised manuscript received 17 June 1985)

We present a cluster growth model for trees (random aggregates without loops). The intrinsic dimension \( d_t \) and fractal dimension \( d_f \) are adjustable. We study the “skeletons” of trees embedded in \( d = 2 \), and find that the intrinsic dimension of a skeleton is \( d_f = 1 \) for \( d_t < d_f = 1.65 \), and \( d_f = 1 + d_t - d_f \) for \( d_t > d_f \). Thus, for \( 1 < d_t < d_f \) these are finitely ramified, and for \( d_f < d_t < 2 \) infinitely ramified. The possibility that structures are fractals in \( f \) space and compact in \( r \) space is also discussed.

The geometrical and topological characteristics of random aggregates are of great interest. Systems such as percolation clusters and lattice “animals” and diffusion-limited aggregation are studied extensively, especially for the purpose of finding those exponents which characterize the physical properties of the systems.\(^{1-8}\) Recently, the usefulness of the intrinsic exponent \( d_t \), which indicates how the mass \( M \) scales with the chemical distance \( l , M \sim l^{d_t} \), has been recognized.\(^{3,5,8}\) However, the manner in which loops, branches, and dead ends affect physical properties of random aggregates still is not clear. Thus, in order to isolate the effects of branches and dead ends from those of loops, we introduce in this Rapid Communication a cluster growth model for trees (clusters without loops) for which \( d_t \) is adjustable, and study their fractal dimension. Recently, also, the concept of the “skeleton” has been introduced.\(^{9}\) The skeleton of a cluster is defined as the ensemble of sites belonging to the shortest paths from a chosen site to the sites in a shell at chemical distance \( L \). Skeleton substructures are important in characterizing physical properties such as diffusion, elasticity, and resistivity.

An important parameter for characterizing skeletons is the exponent \( d_f \), defined in the limit that \( L \to \infty \), and calculated through \( M_f \sim l^{d_f} \), where \( M_f \) is the mass of the skeleton lying within the \( f \)th shell. It has been found\(^9\) that the skeletons of percolation clusters at criticality are chemically linear for all \( d_t \), i.e., \( d_f = 1 \), which provides a quantitative measure for the qualitative property that a structure be finitely ramified. The question arises\(^8\) whether there exist random tree structures which are infinitely ramified, for which \( d_f > 1 \). The growth model which we now consider enables us to generate planar trees which can have any arbitrarily chosen value of \( d_t \). We study properties of the skeletons of these trees and find \( d_f > 1 \) if \( d_t \) is greater than a critical value \( d_f = 1.65 \). Also, we infer that it is possible to obtain compact random structures in real space \( r \) which are fractals in the chemical-distance \( f \) space.

The general model is as follows. We choose a site on a \( d \)-dimensional lattice as the seed of the tree and then select, randomly, \( B(1) \) nearest neighbors of the seed to be occupied. The other nearest-neighbor sites are taken to be blocked. These occupied sites represent the first chemical shell, and their chemical distance to the seed is \( f = 1 \). Similarly, we grow the next shell by examining the nonoccupied and nonblocked nearest neighbors of the sites in the first shell. Of the latter, \( B(2) \) are then randomly occupied, the sites being chosen with the restriction that a new site can be occupied only if it has but one nearest-neighbor already occupied site. (If the site has more than one nearest-neighbor occupied site, it is regarded as blocked.) This restriction

![Trees](a.png)

![Trees](b.png)

![Trees](c.png)

FIG. 1. Trees grown by the cluster growth procedure described in the text \((L = l_{\text{max}} = 300)\). (a) \( d_t = 1.3 \), (b) \( d_t = 1.5 \), (c) \( d_t = 1.8 \).

Work of the U.S. Government

Not Subject to U.S. Copyright
prevents the occurrence of loops in the clusters. Higher-order shells are grown in a similar manner. In order to generate figures of predetermined dimensionality, we choose

\[ B(l) = B_0 l^\alpha, \]

(1a)

from which it follows that \( M(l) \) is given as

\[ M(l) = \sum_{l'=0}^{l} B(l') = \sum_{l'=0}^{l'} \sum_{l=0}^{l'} B(l') = \sum_{l'=0}^{l'} B(l') = \sum_{l'=0}^{l'} l^\alpha = \frac{1}{\alpha+1} l^{\alpha+1}, \]

(1b)

where \( \alpha \) is the intrinsic dimension of the cluster.

In Fig. 1 we show trees grown in \( d = 2 \) with \( \alpha = 1.3, 1.5, \) and \( 1.8 \). The fractal dimension \( d_f \) of these trees was calculated according to the following scheme. Since \( M = \frac{1}{\alpha+1} l^{\alpha+1} \), it follows that \( R = \frac{l^{\alpha+1}}{d_f} \). Thus we numerically evaluate the radius of gyration \( R(l) \) of the clusters,

\[ R(l) = \frac{l^{\alpha+1}}{d_f}, \]

(2)

and use the identity \( \tilde{R} = d_f/d_f \). Results for \( \tilde{R} \) and \( d_f \) for different values of \( \alpha \) are given in Table I. The results for \( \alpha > 1.8 \) are especially interesting. For example, the fact that \( d_f \approx 2.0 \) for \( \alpha = 1.9 \) suggests that a cluster can appear to be compact in the geometrical space \( R \), even though it is fractal in the chemical-distance space \( l \). This distinction is important when considering various transport properties.

![Image](a) (b) (c)

**FIG. 2.** Skeletons of the structures shown in Fig. 1. (a) \( d_f = 1.3 \), (b) \( d_f = 1.5 \), (c) \( d_f = 1.8 \). For \( d_f \leq 1.65 \) the trees are infinitely ramified; i.e., \( d^*=1 \), as is evident in (a) and (b), in which branching occurs only for \( l > l \) where "end effects," caused by proximity to the boundary shell \( L = 300 \), are significant. For \( d_f \geq 1.65 \) the trees are infinitely ramified, i.e., \( d^* > 1 \). Branching occurs almost immediately and continuously for \( l > 1 \) (see also Fig. 3).

The skeletons of the clusters presented in Fig. 1 are shown in Fig. 2. The structures shown in Figs. 2(a) and 2(b) \((d_f = 1.3 \) and \( 1.5 \), respectively\) do not branch appreciably until \( I \) is relatively close to \( L \), whereas in Fig. 2(c) \((d_f = 1.8)\) we observe that branching occurs for values of \( I < L \). The number of sites \( B(I) \) in the \( I \)th shell of such skeletons, averaged over 100 samples, is shown in Fig. 3. We calculated the intrinsic dimension \( d_f \) from these and other curves, and the numbers thus obtained are given in Table I and Fig. 4. It is interesting to compare our results

**FIG. 3.** \( \log B(I) \) vs \( \log I \), where \( B(I) \) is the expected number of sites in a skeleton as a function of the chemical distance \( I \) from the seed (see text). Two types of behavior are indicated: \( d_f = 1.5 < d_f^* \) and \( d_f = 1.9 > d_f^* \), where \( d_f^* = 1.65 \pm 0.05 \) is the critical value of \( d_f \), below which trees are infinitely ramified and above which they are infinitely ramified. (For convenience, the value of \( B_0 \) here was taken to be 4. Results do not differ significantly when \( B_0 = 1 \) [see Eq. (1a)]. Circles, \( L = 200 \); triangles, \( L = 400 \); squares, \( L = 600 \). The lowest curve (diamonds) corresponds to \( d_f = 1.5 \), \( L = 600 \), \( B_0 = 1 \). Note that up to \( I = 300 \) one observes that \( B(I) = 1 \), which substantiates our argument that in this case the skeleton is linear \((d_f^* = 1)\).
for $d_f$ with the result for percolation. Here, for $d_i \leq 1.65$, we obtain $d_f = 1$ as found for percolation clusters at criticality; however, for $d_i \geq 1.65$ we find $d_f > 1$. The exponent $d_f - 1$ characterizes the exponent of ramification of the cluster, thus, for $d_f \leq 1.65$ these trees are finitely ramified (the ramification exponent is zero), and for $d_i \geq 1.65$ the trees are infinitely ramified.

Recently it has been shown, using analytical methods, that similar structures generated on a Cayley tree yield $d_f = 1$ for $d_i \leq 2$ and $d_f = d_i - 1$ for $d_i \geq 2$. These results, together with the results as represented in Fig. 4, are consistent with the following relationship between $d_i$ and $d_f$,

$$d_f = 1 + d_i - d_f$$

for $d_i \leq 1.65$ and

$$d_f = 1$$

for $d_i \geq 1.65$,

where $d_f$ is a critical value. Because Eq. (3), with $d_f = 2$, has been proven analytically for trees grown on a Cayley lattice (which represents growth in high dimensions) and because it seems to hold numerically, with $d_f = 1.65$, for $d = 2$, we surmise that it may be valid (with appropriate $d_f$) for trees constructed in any dimension.

We note that $d_f = 2$ is equal to the value of the intrinsic dimensionality $d_i$ of percolation clusters grown on a Cayley tree. Similarly, the value inferred in the present study, $d_f = 1.65 \pm 0.05$, is close to the value $d_i = 1.64$ which characterizes incipient percolation clusters in $d = 2$. Thus, we conjecture that $d_f$ in our tree-growth model is equal to $d_i$ for critical percolation clusters generated in the same dimension.

In summary, we have presented a tree-growth model for which the intrinsic dimension $d_i$ can be varied. We find that, for $d_i \leq 1.65$, the skeletons of the clusters are linear (i.e., $d_f = 1$) and, for $d_i > 1.65$, the skeletons are nonlinear (i.e., $d_f > 1$). It would be interesting to study the skeletons of other tree-like models such as diffusion-limited aggregation and screened growth aggregates.

---

9Permanent address: Department of Physics, Bar Ilan University, Ramat-Gen, Israel.
22The asymptotic linear portion of the curves in Fig. 3, for which $B(\theta) = \theta^{d_f - 1}$, becomes more prominent as $L$ increases. Consequently, the larger the lattice size, the more accurate will be the fitted value of $d_f$. Here, exponents have been determined from curves for which $L = 600$. Also the accuracy of the deduced value of $d_f$ increases the farther $d_i$ is from $d_f$ again because the linear portion of the curve is more prominent in such an instance.