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Cluster growth model for treelike structures

S. Havlin* and R. Nossal
Physical Sciences Laboratory, Division of Computer Research Technology, National Institutes of Health, Bethesda, Maryland 20205

B. Trus
Computer Systems Laboratory, Division of Computer Research Technology, National Institutes of Health, Bethesda, Maryland 20205

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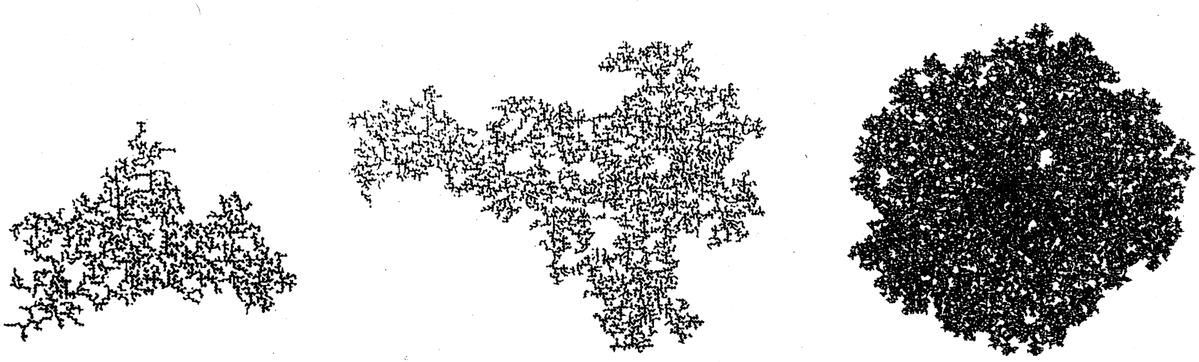
We present a cluster growth model for trees (random aggregates without loops). The intrinsic dimension d_l and fractal dimension d_f are adjustable. We study the "skeletons" of trees embedded in $d=2$, and find that the intrinsic dimension of a skeleton is $d_l^s=1$ for $d_l \leq d_f^c \approx 1.65$, and $d_l^s \approx 1+d_l-d_f^c$ for $d_l \geq d_f^c$. Thus, for $1 \leq d_l \leq d_f^c$ these trees are finitely ramified, and for $d_f^c < d_l \leq 2$ infinitely ramified. The possibility that structures are fractals in l space and compact in r space is also discussed.

The geometrical and topological characteristics of random aggregates are of great interest. Systems such as percolation clusters and lattice "animals" and diffusion-limited aggregation are studied extensively, especially for the purpose of finding those exponents which characterize the physical properties of the systems.[1-8] Recently, the usefulness of the intrinsic exponent d_l, which indicates how the mass M scales with the chemical distance l, $M \sim l^{d_l}$, has been recognized.[3,5,8] However, the manner in which loops, branches, and dead ends affect physical properties of random aggregates still is not clear. Thus, in order to isolate the effects of branches and dead ends from those of loops, we introduce in this Rapid Communication a cluster growth model for trees (clusters without loops) for which d_l is adjustable, and study their fractal dimension. Recently, also, the concept of the "skeleton" has been introduced.[9] The skeleton of a cluster is defined as the ensemble of sites belonging to the shortest paths from a chosen site to the sites in a shell at chemical distance L. Skeleton substructures are important in characterizing physical properties such as diffusion, elasticity, and resistivity.

An important parameter for characterizing skeletons is the exponent d_l^s, defined in the limit that $L \rightarrow \infty$, and calculated through $M_s \sim l^{d_l^s}$, where M_s is the mass of the skeleton lying within the lth shell. It has been found[9] that the skeletons of percolation clusters at criticality are chemically linear for all d, i.e., $d_l^s=1$, which provides a quantitative measure for the qualitative property that a structure be finitely ramified. The question arises[8] whether there exist random tree structures which are infinitely ramified, for which $d_l^s>1$. The growth model which we now consider enables us to generate planar trees which can have any arbitrarily chosen value of $d_l \leq 2$. We study properties of the skeletons of these trees and find $d_l^s>1$ if d_l is greater than a critical value $d_f^c \approx 1.65$. Also, we infer that it is possible to obtain compact random stuctures in real space r which are fractals in the chemical-distance l space.

The general model is as follows. We choose a site on a d-dimensional lattice as the seed of the tree and then select, randomly, $B(1)$ nearest neighbors of the seed to be occupied. The other nearest-neighbor sites are taken to be blocked. These occupied sites represent the first chemical shell, and their chemical distance to the seed is $l=1$. Similarly, we grow the next shell by examining the nonoccupied and nonblocked nearest neighbors of the sites in the first shell. Of the latter, $B(2)$ are then randomly occupied, the sites being chosen with the restriction that a new site can be occupied only if it has but one nearest-neighbor already occupied site. (If the site has more than one nearest-neighbor occupied site, it is regarded as blocked.) This restriction



(a) (b) (c)

FIG. 1. Trees grown by the cluster growth procedure described in the text ($L=l_{max}=300$). (a) $d_l=1.3$, (b) $d_l=1.5$, (c) $d_l=1.8$.

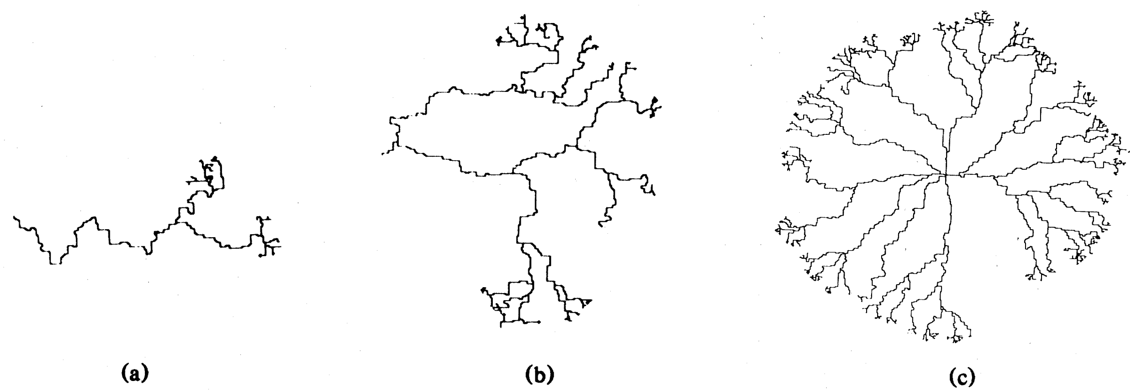


FIG. 2. Skeletons of the structures shown in Fig. 1. (a) $d_l = 1.3$, (b) $d_l = 1.5$, (c) $d_l = 1.8$. For $d_l \lesssim 1.65$ the trees are finitely ramified; i.e., $d_l^s = 1$, as is evident in (a) and (b), in which branching occurs only for $l >> 1$ where "end effects," caused by proximity to the boundary shell $L = 300$, are significant. For $d_l \gtrsim 1.65$ the trees are infinitely ramified, i.e., $d_l^s > 1$. Branching occurs almost immediately and continuously for $l > 1$ (see also Fig. 3).

prevents the occurrence of loops in the clusters. Higher-order shells are grown in a similar manner. In order to generate figures of predetermined dimensionality, we choose

$$B(l) = B_0 l^{\alpha} , \tag{1a}$$

from which it follows that $M(l)$ is given as

$$M(l) = \sum_{l'=1}^{l} B(l') \sim l^{\alpha+1} = l^{d_l} , \tag{1b}$$

where d_l is the intrinsic dimension of the cluster.

In Fig. 1 we show trees grown in $d = 2$ with $d_l = 1.3$, 1.5, and 1.8. The fractal dimension d_f of these trees was calculated according to the following scheme. Since $M \sim l^{d_l} \sim R^{d_f}$, it follows that $R \sim l^{d_l/d_f}$. Thus we numerically evaluate the radius of gyration $R(l)$ of the clusters,

$$R(l) \sim l^{\tilde{\nu}} , \tag{2}$$

and use the identity $\tilde{\nu} = d_l/d_f$. Results for $\tilde{\nu}$ and d_f for different values of d_l are given in Table I. The results for $d_l > 1.8$ are especially interesting. For example, the fact that $d_f \approx 2.0$ for $d_l = 1.9$ suggests that a cluster can appear to be compact in the geometrical space r, even though it is fractal in the chemical-distance space l. This distinction is important when considering various transport properties.[8,11]

TABLE I. Exponents of trees and skeletons. Calculated values for the fractal dimensionality d_f, ratio $\tilde{\nu} = d_l/d_f$, and skeleton intrinsic dimensionality d_l^s, for clusters having intrinsic dimensional d_l. The error bars were determined by calculating the extreme slopes when fitting linear curves to the corresponding quantities.

d_l	d_f	$\tilde{\nu}$	d_l^s
1.3	1.71 ± 0.03	0.76 ± 0.02	$1.0^{+0.03}_{-0.0}$
1.5	1.82 ± 0.03	0.82 ± 0.02	$1.0^{+0.03}_{-0.0}$
1.6	1.86 ± 0.03	0.86 ± 0.02	$1.0^{+0.5}_{-0.0}$
1.7	1.91 ± 0.03	0.89 ± 0.02	1.1 ± 0.05
1.8	1.95 ± 0.03	0.92 ± 0.02	1.2 ± 0.05
1.9	$2.0^{+0}_{-0.03}$	0.95 ± 0.02	1.27 ± 0.04
2.0	$2.0^{+0}_{-0.03}$	$1^{+0.0}_{-0.02}$	1.37 ± 0.03

The skeletons of the clusters presented in Fig. 1 are shown in Fig. 2. The structures shown in Figs. 2(a) and 2(b) ($d_l = 1.3$ and 1.5, respectively) do not branch appreciably until l is relatively close to L, whereas in Fig. 2(c) ($d_l = 1.8$) we observe that branching occurs for values of $l << L$. The number of sites $B^s(l)$ in the lth shell of such skeletons, averaged over 100 samples, is shown in Fig. 3. We calculated the intrinsic dimension d_l^s from these and other curves, and the numbers thus obtained are given in Table I and Fig. 4. It is interesting to compare our results

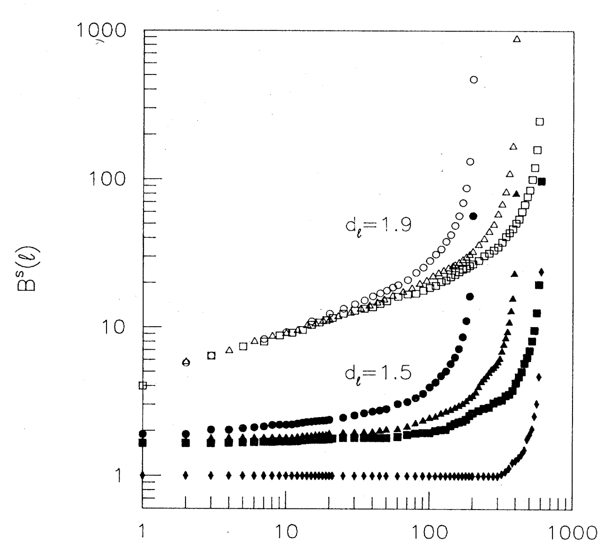


FIG. 3. $\log B^s(l)$ vs $\log l$, where $B^s(l)$ is the expected number of sites in a skeleton as a function of the chemical distance l from the seed (see text). Two types of behavior are indicated: $d_l = 1.5 < d_l^c$ and $d_l = 1.9 > d_l^c$, where $d_l^c = 1.65 \pm 0.05$ is the critical value of d_l, below which trees are finitely ramified and above which they are infinitely ramified. (For convenience, the value of B_0 here was taken to be 4. Results do not differ significantly when $B_0 = 1$ [see Eq. (1a)]. Circles, $L = 200$; triangles, $L = 400$; squares, $L = 600$. The lowest curve (diamonds) corresponds to $d_l = 1.5$, $L = 600$, $B_0 = 1$. Note that up to $l \simeq 300$ one observes that $B^s(l) = 1$, which substantiates our argument that in this case the skeleton is linear ($d_l^s = 1$).

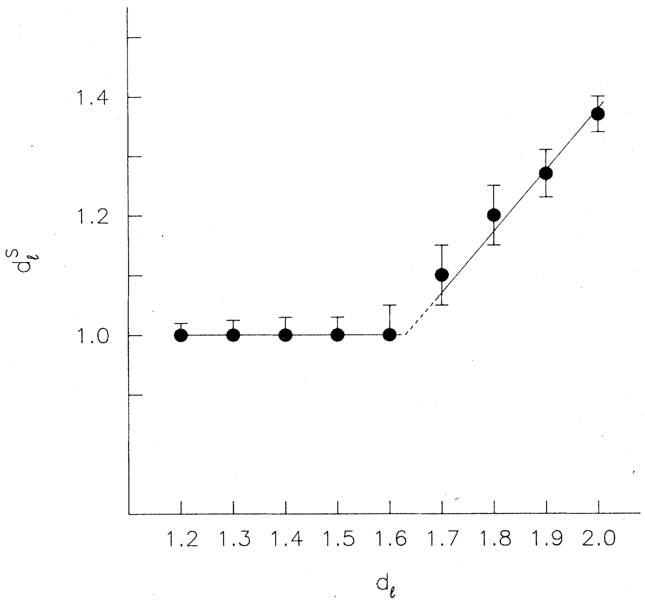


FIG. 4. Skeleton intrinsic dimensionality d_l^s for clusters having intrinsic dimensionality d_l. The critical dimensionality is believed to be approximately $d_l^c = 1.65 \pm 0.05$.

for d_l^s with the result for percolation. Here, for $d_l \lesssim 1.65$, we obtain $d_l^s = 1$ as found for percolation clusters at criticality;[9] however, for $d_l \gtrsim 1.65$ we find $d_l^s > 1$. The exponent $d_l^s - 1$ characterizes the exponent of ramification of the cluster;[10] thus, for $d_l^c \lesssim 1.65$ these trees are *finitely* ramified (the ramification exponent is zero), and for $d_l \gtrsim 1.65$ the trees are *infinitely* ramified.

Recently it has been shown,[12] using analytical methods, that similar structures generated on a Cayley tree yield $d_l^s = 1$ for $d_l \leqslant 2$ and $d_l^s = d_l - 1$ for $d_l \geqslant 2$. These results, together with the results as represented in Fig. 4, are consistent with the following relationship between d_l and d_l^s,

$$d_l^s = 1 \quad \text{for } d_l \leqslant d_l^c ,$$
$$d_l^s = 1 + d_l - d_l^c \quad \text{for } d_l \geqslant d_l^c , \tag{3}$$

where d_l^c is a critical value.[13] Because Eq. (3), with $d_l^c = 2$, has been proven analytically for trees grown on a Cayley lattice (which represents growth in high dimensions) and because it seems to hold numerically, with $d_l^c \approx 1.65$, for $d = 2$, we surmise that it may be valid (with appropriate d_l^c) for trees constructed in any dimension.

We note that $d_l^c = 2$ is equal to the value of the intrinsic dimensionality d_l of percolation clusters grown on a Cayley tree. Similarly, the value inferred in the present study, $d_l^c = 1.65 \pm 0.05$, is close to the value $d_l = 1.64$ which characterizes incipient percolation clusters in $d = 2$.[3] Thus, we conjecture that d_l^c in our tree-growth model is equal to d_l for critical percolation clusters generated in the same dimension.

In summary, we have presented a tree-growth model for which the intrinsic dimension d_l can be varied. We find that, for $d_l \leqslant d_l^c$, the skeletons of the clusters are linear (i.e., $d_l^s = 1$) and, for $d_l > d_l^c$, the skeletons are nonlinear (i.e., $d_l^s > 1$). It would be interestng to study the skeletons of other treelike models such as diffusion-limited aggregation[14] and screened growth aggregates.[15]

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*Permanent address: Department of Physics, Bar Ilan University, Ramat-Gen, Israel.

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[13]The asymptotic linear portion of the curves in Fig. 3, for which $B^s(l) \sim l^{d_l^s - 1}$, becomes more prominent as L increases. Consequently, the larger the lattice size, the more accurate will be the fitted value of d_l^s. Here, exponents have been determined from curves for which $L = 600$. Also the accuracy of the deduced value of d_l^s increases the farther d_l is from d_l^c, again because the linear portion of the curve is more prominent in such an instance.

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