Anomalous logarithmic slow-dynamics behavior on hierarchical and random systems

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We study diffusion on hierarchical and random potential-well and barrier structures. In the case of a power-law distribution of barrier size for $d=1$ where the transition rate follows a Boltzmann distribution, we find ultraslow diffusion characterized by logarithmic time-dependent displacements. This result is valid for both barriers and wells. For $d \geq 2$ dimensions, we find a universal slow logarithmic dependence for the case of wells independent of d. For the case of barriers, we find that the logarithmic anomalous transport depends on the dimension of the system.

I. INTRODUCTION

The dynamics in one-dimensional systems have been studied since the classical paper by Dyson.[1] The original problem had been mapped onto several fields.[2-3] The diffusion interpretation of Dyson's equation was extensively studied mainly by solving the complicated integral equation or via the n-replica spin Hamiltonian.[4] The general diffusion equation can be written in matrix form as

$$C_r P_r = W P_r \; , \tag{1}$$

where P_r denotes the probability of finding a particle at time t at site r; C_r denotes the trapping rate at site r and the transition matrix W is a Jacobian matrix allowing hopping to nearest-neighbors (NN) sites, assuming symmetrical transition rates. The elements W_{rs} and C_r are random variables with given distributions. The elements C_r represent the mass value, and W_{rs} the harmonic force constants in Dyson's equation. Here we study two cases.

(i) W_{rs} is random and C_r is an arbitrary constant, therefore, $T_{r,r+1} = T_{r+1,r}$, where T denotes the effective transition rate, i.e., W_{rs} divided by C_r.

(ii) W_{rs} is an arbitrary constant and C_r is random, therefore, $T_{r,r+1} = T_{r,r-1}$.

The first case is known as random barriers (or hopping) and the second is known as random wells (or trapping). Dyson[1] showed that in one dimension these two cases reduce to the same type of integral equation, and hence to similar results. In the following we denote T_r by W_r. The solution of the Dyson equation shows that there exist two classes, one for which $\langle W^{-1} \rangle$ is finite, and the second for which $\langle W^{-1} \rangle$ diverges. For the first class, the solution does not differ essentially from the solution of the ordered chain, whereas solving the second class is a very complicated mathematical problem which is generally not solvable. Indeed, concerning a power-law distribution $P(W) \sim W^{-\alpha}$, $0<\alpha<1$, it was shown that the probability $P_0(t)$ of being at the origin and the mean-square displacement $\langle x^2 \rangle$ are both anomalous of the form $P_0(t) \sim t^{-1/d_w}$ and $\langle x^2 \rangle \sim t^{2/d_w}$, where $d_w = (2-\alpha)/(1-\alpha)$. Using the general Einstein relation between conductivity and diffusion,[5] anomalous behavior is found for the dynamical conductivity. This solution was found relevant to various physical systems such as quasilocalization found in some one-dimensional (1D) superionic conductors,[6] anomalous relaxation in spin systems,[7] and biased diffusion in random structures.[8]

In this work we present a general scaling-type argument to solve $\langle x^2 \rangle$ in the 1D general case where $\langle W^{-1} \rangle$ diverges. We apply this method specifically to the marginal case where $\alpha = 1$. In particular, we study systems with a power-law distribution of potential barriers or wells where the transition rates are given by Boltzmann factors. We find in this case ultraslow diffusion characterized by a displacement varying as a power of $\ln(t)$ rather than of t, and that the conductance decreases exponentially with the size of the system. Moreover, since in 1D systems only $\langle W^{-1} \rangle$ is important, or specifically how $\langle W^{-1} \rangle$ diverges, we suggest that W be arranged in a certain order, say hierarchically, and use an exact real-space renormalization technique to find the flow line away from the fixed point.

In higher dimensions $D \geq 2$, we find that the transport properties of barriers is different from those of wells. The reason for this is that the particle in the wells hops to all its nearest-neighbor wells with equal probability regardless of the well depth. On the other hand, in the case of nearest-neighbor barriers, the particle has a larger probability to hop over the lowest nearest-neighbor barrier. Therefore, for the case of wells, the particle might be trapped in deeper wells more often for higher dimension, whereas for barriers, the high barriers are systematically avoided.

The barrier problem is mapped to n parallel stripes of random conductors in d dimensions. We assume an anisotropic model in the sense that the vertical conductors are perfect. If the stripes are combined using Kirchoff's law, we find that the conductance exponent depends on the dimension of the system.

For the case of wells, we find for $d \geq 2$ a universal slow logarithmic dependence, independent of d, i.e., $d=2$ is the upper critical dimension of the problem.

This family of transport laws exhibits the "freezing" of the dynamics. A similar case treated previously gave rise to an interesting relaxation law in spin-glass systems.[9]

In Sec. II, we present a general hierarchical model. Using the renormalization technique, we find the diffusion law and the autocorrelation function in this system. We study the corresponding problem with randomly distributed wells and show the connection between the two systems. The analogue problem of randomly distributed conductors is also examined.

In Sec. III we present a general treatment for systems in $d \geq 2$. For the well system, we find a slow logarithmic diffusion law, independent of the dimension. For the conductor and barrier systems we show that the logarithmic transport law depends on the dimension of the system.

II. ONE-DIMENSIONAL SYSTEMS

A. Hierarchical potential wells

Consider the following hierarchical potential well structure (see Fig. 1). The depth of the kth well Δ_k is

$$\Delta_k = R^l, \quad R > 1 \tag{2a}$$

where l is determined by

$$k(\mathrm{mod}\, 2^l) = 2^{l-1}, \quad l > 0, \text{ integer} . \tag{2b}$$

The transition rate between two neighboring wells, k and $k \pm 1$, is assumed to be given by the Boltzmann factor

$$W_{k,k\pm 1} = e^{-\beta\Delta_k} . \tag{3}$$

Defining new renormalized cells n according to $k = 2^{n+1}$, we can write a recursion relation for the transition rates

$$\frac{1}{W_n} = \frac{2}{W_{n-1}} + e^{\beta R^n} , \tag{4}$$

where

$$\frac{1}{W_n} = \sum_{k=1}^{2^{n+1}-1} \frac{1}{W_{k,k+1}} . \tag{5}$$

Following the general formalism of Zwanzig,[10] the diffusion constant for $t \to \infty$ can be written as

$$\frac{t}{\langle x^2 \rangle} \equiv D^{-1} = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{W_{k,k+1}} , \tag{6}$$

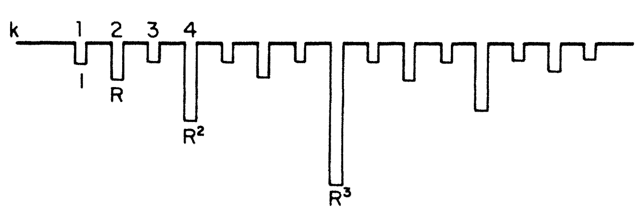

FIG. 1. Hierarchical well structure. The particle can hop from one well to the nearest-neighbor well with a transition rate given by $W_{k,k+1} = e^{-\beta\Delta_k}$ where $\Delta_k = R^l$ is the depth of the kth well.

where N is the number of distinct sites visited by the hopping particle. Equations (4)–(6) lead to

$$D^{-1} = \frac{1}{2} \sum_{n=0}^{n_{\max}} \frac{e^{\beta R^n}}{2^n} , \tag{7}$$

where the upper limit $n_{\max}$ scales as $\langle x^2 \rangle^{1/2} \sim 2^{n_{\max}+1}$. Equation (7) can be written as

$$D^{-1} = \frac{1}{2} \sum_{n=0}^{n_{\max}} \sum_{l=0}^{\infty} \frac{(\beta R^n)^l}{2^n l!}$$
$$= \frac{1}{2} \sum_{l=0}^{\infty} \frac{\beta^l}{l!} \sum_{n=0}^{n_{\max}} \left[\frac{R^l}{2} \right]^n \simeq \frac{e^{\beta R^{n_{\max}}}}{2^{n_{\max}+1}} . \tag{8}$$

Thus, the main contribution to the sum comes from the deepest well. In the limit of large displacement, $\langle x^2 \rangle^{1/2}$, i.e., $n_{\max} \gg 1$, Eq. (8) yields

$$\langle x^2 \rangle \sim (\ln t)^{2\ln 2/\ln R} . \tag{9}$$

For the general case of hierarchical structure generated by an ultrametric tree with coordination number z, Eq. (4) becomes

$$\frac{1}{W_n} = \frac{z}{W_{n-1}} + (z-1)e^{\beta R^n} \tag{10}$$

and

$$n_{\max} \sim \langle x^2 \rangle^{1/2} \sim z^{k+1} . \tag{11}$$

Therefore,

$$\langle x^2 \rangle \sim (\ln t)^{2\ln z/\ln R} . \tag{12}$$

Thus, the diffusion is anomalously slow, characterized by a logarithmic dependence for the displacement. Using scaling arguments[11] we can derive $P_0(t)$. We expect $P_0(t) \sim \langle x^2 \rangle^{-1/2}$ and therefore $P_0(t) \sim (\ln t)^{-\ln 2/\ln R}$.

B. Random potential wells

In the following we compare the above results with the problem of diffusion in a one-dimensional system with randomly distributed wells[12] whose depth Δ is distributed as

$$P(\Delta) \sim \Delta^{-(1+\gamma)} . \tag{13}$$

The transition rates are $W_{k,k+1} \sim e^{-\beta\Delta k}$ from which it follows that their distribution is

$$P(W) \sim P(\Delta)\frac{d\Delta}{dW} \sim \frac{1}{W |\ln W|^{1+\gamma}} . \tag{14}$$

Applying the same formalism as in (6), we obtain

$$D^{-1} = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{W_{k,k+1}} = \int_{W_{\min}}^{W_0} \frac{dW}{W^2 |\ln W|^{1+\gamma}} . \tag{15}$$

The lower limit $W_{\min}$ is the lowest transition rate within a system of N sites and approaches to zero as $t \to \infty$, or $\langle x^2 \rangle \to \infty$. In order to obtain the asymptotic behavior of the diffusion, one must know how $W_{\min}$ scales with $\langle x^2 \rangle$. The upper limit which corresponds to the case of the shallow wells does not contribute to the integral, but rather deals with the case of normal diffusion or conduction.

Evaluating the main contribution to the integral, we find

$$D^{-1} \sim [W_{\min} \mid \ln W_{\min} \mid^{1+\gamma}]^{-1} . \tag{16}$$

Averaging over different initial conditions corresponds to the calculation of the average value of $W_{\min}$. This is done by calculating $\Delta_{\max}$ which is the average maximum depth of a well in a system of size $X \equiv \langle x^2 \rangle^{1/2}$. We choose a homogeneous random variable v so that $P(\Delta) d\Delta \sim dv$ from which follows $v \sim \Delta^{-\gamma}$. Since v is homogeneous, the average of $v_{\min} \sim 1/X$ and thus $\Delta_{\max} \sim X^{1/\gamma}$ and $W_{\min} \sim \exp(-X^{1/\gamma})$. Thus, for $X \gg 1$, we obtain

$$D^{-1} = t/X^2 \sim \exp(X^{1/\gamma}) \tag{17}$$

or, asymptotically,

$$\langle x^2 \rangle \sim (\ln t)^{2\gamma} . \tag{18}$$

Here, the main contribution to the integral is from the lower limit $W_{\min}$ which corresponds to the deepest well as in the case of hierarchical wells. These two cases of hierarchical and random wells coincide in one dimension as expected, because γ can be related[13] to R by $\gamma = \ln R / \ln z$.

The analogue of the distribution of conductors σ

$$P(\sigma) \sim \frac{1}{\sigma \mid \ln\sigma \mid^{1+\gamma}} \tag{19}$$

can be calculated using a similar approach. The conduction Σ is given by

$$\Sigma^{-1} = \sum_{i=1}^{L} \frac{1}{\sigma_i} \sim L \int_{\sigma_{\min}}^{\sigma_0} \frac{P(\sigma)}{\sigma} d\sigma . \tag{20}$$

Evaluating this integral yields

$$\Sigma^{-1} \sim \exp(L^{1/\gamma}) . \tag{21}$$

III. TRANSPORT IN $d \geq 2$ SYSTEMS

A. Potential wells

To study transport in a $d \geq 2$ system with the distribution of transition probabilities given by (14), we use (6) where N is the number of distinct sites visited by the hopping particle. The validity of this equation for higher dimensions has been justified analytically[14] and numerically.[15] The difference between one-dimensional case and higher dimensions appears only in the calculation of $W_{\min}$. As before, we choose v to be a homogeneous random variable but here $v_{\min} \sim 1/\langle x^2 \rangle$, since the number of distinct sites visited in $d \geq 2$ is proportional to $\langle x^2 \rangle$. Hence $\Delta_{\max} \sim \langle x^2 \rangle^{1/\gamma}$ and $W_{\min} \sim \exp[-\langle x^2 \rangle^{1/\gamma}]$. Therefore

$$\langle x^2 \rangle \sim (\ln t)^{\gamma}, \quad 0 < \gamma < \infty, \quad d \geq 2 . \tag{22}$$

Thus, we find a slower diffusion rate than for one dimension. This can be explained by a bigger chance of the particle being "caught" in a deeper well, since it visits more distinct sites than in the one-dimensional case. Note that $d = 2$ is the upper critical dimension since the number of distinct sites for all dimensions $d \geq 2$ has the same scaling with $\langle x^2 \rangle$. Since $P_0(t) \sim \langle x^2 \rangle^{-1/2}$, the appropriate autocorrelation function will be $P_0(t) \sim (\ln t)^{-\gamma/2}$.

B. Conductors

The problem of transport in $d \geq 2$ dimensions with a conductivity distribution given by Eq. (19) is more complicated. Here we have to deal with two difficulties that do not exist in the case of the wells. In the wells system the particle has the same probability to enter any well independent of its depth, where in the present case, the current will flow mainly through the good conductors. Secondly, here we cannot use Eq. (6) which is based on the number of sites visited by a walker because the physical laws are now different. Therefore, we simplify the problem by first solving the problem of n connected rows of conductors and assume that the vertical conductors are perfect (see Fig. 2). This implies that we get only a lower bound for the conductivity exponent. Using Kirchoff's law for n rows of conductors, one obtains

$$\Sigma_n^{-1} = L \int_{\sigma_{\min}}^{\sigma_0} \underset{n}{\cdots} \int_{\sigma_{\min}}^{\sigma_0} \frac{\prod_{i=1}^{n} \frac{d\sigma_i}{\sigma_i \mid \ln\sigma_i \mid^{1+\gamma}}}{\sum_{i=1}^{n} \sigma_i} . \tag{23}$$

Since

$$\left[\sum_{i=1}^{n} \frac{1}{\sigma_i} \right]^{-1} = \int_0^{\infty} \exp\left[-u \sum_{i=1}^{n} \sigma_i \right] du , \tag{24}$$

we can rewrite (23) as

$$\Sigma_n^{-1} = L \int_0^{\infty} du \left[\int_{\sigma_{\min}}^{\sigma_0} \frac{\exp(-u\sigma) d\sigma}{\sigma \mid \ln\sigma \mid^{1+\gamma}} \right]^n$$

$$\equiv L \int_0^{\infty} du\, I^n , \tag{25}$$

where

$$I = \int_{\sigma_{\min}}^{\sigma_0} \frac{\exp(-u\sigma) d\sigma}{\sigma \mid \ln\sigma \mid^{1+\gamma}} . \tag{26}$$

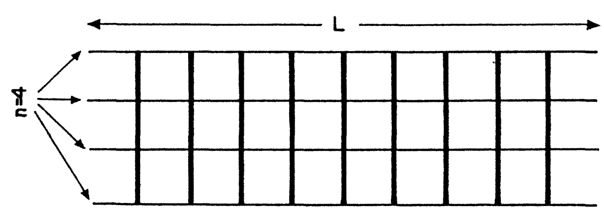

FIG. 2. A quasi-one-dimensional system of length L with $n = 4$ connected linear chains. The vertical bonds are perfect conductors, whereas the horizontal bonds are randomly distributed according to Eq. (14).

Equation (25) can be written as

$$\Sigma_n^{-1}=L\int_0^{\infty} e^{n\ln I}du \ . \tag{27}$$

For large n, we can apply the method of steepest descent.

It can be shown that the main contribution to the integral I [Eq. (26)] comes from the $u\rightarrow 0$ limit. Calculating the integral I appearing in (25) by parts yields

$$I\simeq\frac{-\exp(-u\sigma_{\min})}{\gamma\,|\ln\sigma_{\min}|^{\gamma}}+\frac{u}{\gamma}\int_{\sigma_{\min}}^{\sigma_0}\frac{\exp(-u\sigma)d\sigma}{|\ln\sigma|^{\gamma}} \ . \tag{28}$$

The second integral can be written as

$$\int_0^{-\ln\sigma_{\min}}\frac{e^{-x}\exp(-ue^{-x})}{x^{\gamma}}dx=\sum_{l=0}^{\infty}\frac{(-u)^l}{l!}\int_0^{-\ln\sigma_{\min}}dx\frac{e^{-x(l+1)}}{x^{\gamma}} \tag{29}$$

using the substitution $\sigma=e^{-x}$.

Using the incomplete γ function $\tilde{\gamma}(\nu,\mu u)$, defined by

$$\int_0^u x^{\nu-1}e^{-\mu x}dx\equiv\mu^{-\nu}\tilde{\gamma}(\nu,\mu u),\quad \nu>0 \tag{30}$$

we obtain

$$I\simeq\frac{\exp(-u\sigma_{\min})}{\gamma\,|\ln\sigma_{\min}|^{\gamma}}+\frac{u}{\gamma}\sum_{l=0}^{\infty}\frac{(-u)^l}{l!}(l+1)^{\gamma-1}\tilde{\gamma}(1-\gamma,-(l+1)\ln\sigma_{\min}) \tag{31}$$

for $\gamma<1$.

Since we are interested in the limit $u\rightarrow 0$, we can approximate I by

$$I\simeq\frac{\exp(-u\sigma_{\min})}{\gamma\,|\ln\sigma_{\min}|^{\gamma}} \ , \tag{32}$$

and, therefore,

$$\Sigma^{-1}\sim L\int_0^{\infty}du\, I^n\sim\frac{L}{|\ln\sigma_{\min}|^{\gamma n}n\,\sigma_{\min}\gamma^n} \ . \tag{33}$$

Calculating the average $\sigma_{\min}$ in the case of n connected rows we must demand that $\sigma_{\min}$ scales in the same way for every row having size L. For one row we have shown that

$$\sigma_{\min}\sim\exp(-L^{1/\gamma}) \ . \tag{34}$$

Therefore, for n rows

$$\prod_{i=1}^{n}\sigma_{\min}=\sigma_{\min}^{n}\sim\exp(-L^{1/\gamma}) \tag{35}$$

and, hence,

$$\frac{\Sigma_n^{-1}}{L}\sim\frac{\exp(L^{1/\gamma}/n)}{n^{-\gamma n}L^n n} \ . \tag{36}$$

For the two-dimensional case, $n\simeq L$ and

$$\Sigma_n^{-1}\sim\frac{\exp(L^{1/\gamma-1})}{L^{L(1-\gamma)}} \ . \tag{37}$$

Since

$$\lim_{L\rightarrow\infty}\frac{e^{L^{\delta}}}{L^L}\sim e^{L^{\delta}},\quad \delta>1 \tag{38}$$

we get

$$\Sigma^{-1}\sim\begin{cases}\exp(L^{1/\gamma-1}), & \gamma<\frac{1}{2}\\ L^0, & \gamma\geq\frac{1}{2}\end{cases} \ . \tag{39}$$

For d dimensions we can assume $n\sim L^{d-1}$ and

$$\Sigma^{-1}\sim\frac{\exp(L^{1/\gamma-(d-1)})}{L^{L^{d-1}}[1-\gamma(d-1)]} \ . \tag{40}$$

Hence the anomalous region is given by

$$\gamma<\frac{1}{2(d-1)} \ . \tag{41}$$

Note that for $d=1$, (37) and (39) reduce to the result $\Sigma^{-1}\sim\exp(L^{1/\gamma})$ for $0<\gamma<\infty$ obtained in Sec. II. Using Eq. (6) we find

$$\langle r^2\rangle\sim\begin{cases}t, & \gamma\geq\gamma_c\\ (\ln t)^{2\gamma/[1-\gamma(d-1)]}, & \gamma<\gamma_c\end{cases} \tag{42}$$

where $\gamma_c=1/[2(d-1)]$.

The above results, Eqs. (39) and (42), are rigorous lower bounds for the general case, in which all bonds are distributed according to (14).

In summary, we find localization in all dimensions for diffusion in the presence of a power-law distribution of potential wells and Boltzmann transition rates. We show that the upper critical dimension is $d=2$ with $\langle r^2\rangle\sim(\ln t)^{2\gamma}$ for all $d\geq 2$. For the case of barriers, we find that for any number of dimensions, $\langle r^2\rangle$ scales as power of $\ln(t)$ which depends on the dimension. Moreover, for any number of dimensions, we find a dynamical phase transition[13,16,17] from normal diffusion to logarithmic anomalous diffusion (localization). There is also a critical distribution, characterized by γ_c, below which localization occurs.

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