LETTER TO THE EDITOR

Structure of clusters generated by random walks

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Abstract. We study the cluster structure resulting from a nearest-neighbour random walk embedded in a $d$-dimensional space. Each bond visited by the random walks is regarded as belonging to the cluster. The diffusion exponent $d_\tau$ and the fracton dimensional of the fractal cluster in $d = 3$ is found to be $d_\omega = 3.5 \pm 0.1$ and $\tilde{d} = 0.57 \pm 0.02$, using a method of exact enumeration of random walks on these fractals.

The structure of statistical fractals is currently a subject of intensive study (Mandelbrot 1982, Stanley and Coniglio 1982, Meakin 1983, Alexander and Orbach 1982, Havlin and Nossal 1984). The main properties of interest are the fractal dimensionality of the clusters $d_\tau$, transport properties characterised by diffusion and resistivity exponents ($d_\tau$ and $d_\omega$, respectively) and the fracton dimensionality (Alexander and Orbach 1982) $\tilde{d} = 2d_\tau/d_\omega$. The possible relation between 'static' exponents like $d_\tau$ and a 'dynamic' one like $d_\omega$ is of special interest mostly because of the conjecture (Alexander and Orbach 1982, AO) relating these quantities for percolation clusters and the proposed extension (Meakin and Stanley 1983, Leyvraz and Stanley 1983) of this conjecture to other statistical fractals.

In this work we study the fundamental problem of the fractal structure resulting from a nearest-neighbour random walk (RW) of $N_1$ steps embedded in $d$-dimensional space. Each bond visited by the RW is regarded as a bond belonging to the cluster. The results for the limit $N_1 \to \infty$ in $d = 1, 2$ and 4 are obvious. For $d = 1$ and 2 the RW fills homogeneously all space and therefore one just obtains the homogeneous space with $\tilde{d} = d_\tau = d$. For $d = 4$ it is usually assumed that the RW intersects itself in a number of places which is negligible upon scaling for the purpose of calculating exponents (de Gennes 1979), thus the order of ramification is essentially 2 and the resulting fractal is quasi-linear with $d_\tau = 2$ and $\tilde{d} = 1$. However for the limit $N \to \infty$ in $d = 3$, the intrinsic structure of the fractal resulting from the RW has not so far been established. Thus it is of interest to study the diffusion exponents, $d_\omega$ and $\tilde{d}$, of these fractals. We present scaling arguments as well as numerical data for $d_\omega$ and $\tilde{d}$. Moreover, we relate the results found here for diffusion with the results of some recent work (Banavar et al 1983) on the resistivity of a random walk by the Einstein relation.

We study the diffusion exponent and the fracton dimensionality of a RW cluster by performing a second RW on it. The second RW whose starting point is uniformly distributed over the distinct sites visited by the first RW, is allowed to walk on any

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lattice bond visited by the first walker. Thus the second RW walks in a substrate which is ramified, in contrast to the situation treated in earlier analyses (Kehr and Kutner 1982). Assume that the fractal dimensionality of the RW substrate resulting from the first walker is \( d_f \), \( R^d \) scales as the expected number of distinct sites visited where \( R \) is the linear span of the RW, and that the anomalous diffusion exponent of the second RW is \( d_w \), then

\[
\bar{d} = 2d_f/d_w
\] (1)

is the fraction dimensionality (Alexander and Orbach 1982) of the RW substrate.

We seek the function \( \langle R_2^2(N_1, N_2) \rangle \) giving the end-to-end mean-square distance of an \( N_2 \)-steps RW performed on a substrate resulting from an \( N_1 \)-step RW. We first derive the exact result for the \( d = 1 \) case. In this case

\[
\langle R_2^2(N_1, N_2) \rangle = \int_0^\infty \langle R_2^2(L, N_2) \rangle g(L, N_1) \, dL
\] (2)

where \( g(L, N_1) \) is the probability density for the span \( L \) of the substrate random walk and \( \langle R_2^2(L, N_2) \rangle \) is the mean square end-to-end distance of an \( N_2 \)-step RW performed on the segment of length \( L \). It can be shown that (Weiss and Rubin 1976)

\[
g(L, N_1) = \frac{8}{(4\pi\sigma N_1)^{1/2}} \sum_{j=1}^{\infty} (-j)^{j+1} j^2 \exp[-j^2 L^2/(4\sigma N_1)]]
\] (3)

where \( \sigma \) is the diffusion constant of the first walk. When the integral in equation (2) is evaluated, one finds

\[
\langle R_2^2(N_1, N_2) \rangle \approx \langle (\sigma N_1) \ln 2 - 128/\pi^3 \sigma (N_1 N_2)^{1/2} \frac{1}{(2l+1)^3} \frac{1}{1+\exp[\pi(2l+1)(N_2/N_1)^{1/2}]}
\]

\[
+ \frac{1}{\pi} \frac{1}{(2l+1)^4} \left( \frac{N_1}{N_2} \right)^{1/2} \ln \left( 1+\exp \left[ -\pi(2l+1)\left( \frac{N_2}{N_1} \right)^{1/2} \right] \right) \right)
\] (4)

We see that \( R_1^2 \) is of the form

\[
\langle R_2^2(N_1, N_2) \rangle \approx N_1^a N_2^b f(N_2^c/N_1)
\] (5)

with \( a = \frac{1}{2}, b = \frac{1}{2} \) and \( c = 1 \). We assume this scaling form applies for any dimensionality \( d \) with suitable values of \( a, b, \) and \( c \). In fact the limits of \( f(x = N_2^c/N_1) \) for \( x \to 0 \) and \( x \to \infty \) are easily obtained from simple arguments. For \( x \to \infty \) the number of steps taken by the second walker is much larger than that taken by the first. Its span therefore limits \( \langle R_2^2 \rangle \) to be proportional to \( N_1 \) and independent of \( N_2 \). For \( x \to 0 \) the second walker is with overwhelming probability far from the edges of the fractal substrate resulting from the first RW, therefore \( \langle R_2^2 \rangle \approx N_2^{2/d_w} \) independent of \( N_1 \). Thus, we have

\[
f(x) = \begin{cases} \ x^a & x \to 0 \\ x^{-b/c} & x \to \infty. \end{cases}
\] (6)

Substituting (6) into (5) yields

\[
\langle R_2^2 \rangle \approx \begin{cases} \ x^{b+ca} & N_1 \to \infty, N_2 \text{ finite} \\ x^{(b+ca)/c} & N_1 \text{ finite, } N_2 \to \infty. \end{cases}
\] (7)
Our earlier remarks therefore imply the relations
\[ c = 2/d_w, \quad \frac{1}{2}bd_w + a = 1. \] (8)

Introducing these identities into (5) one gets
\[ \langle R_2^2/N_1 \rangle = (N_2^{2/d_w}/N_1)^{bd_w/2} f(N_2^{2/d_w}/N_1) = g(N_2^{2/d_w}/N_1), \] (9)

where
\[ g(x) \approx \begin{cases} x & x \ll 1 \\ A = \text{constant} & x \gg 1. \end{cases} \] (10)

Thus we see that only a single exponent, \( 2/d_w \), suffices to describe limiting properties of \( \langle R_2^2 \rangle \).

For the cases \( d = 1 \) and \( d = 2 \) we expect, for \( N_1 \rightarrow \infty \), that \( d_w = 2 \) since the substrate of the first walk is the entire space. However, the limit \( g(x) \approx x \), with \( d_w = 2 \), for \( x \ll 1 \) in (10) did not occur in our simulations in \( d = 2 \) even for \( N_1 = 20\,000 \) steps. The \( \text{RW} \) in \( d = 3 \) was studied carefully in the range \( x \ll 1 \) for which \( R_2^d = N_2 \) follows from (9) and (10). In figure 1 we present data for \( d = 3 \) and \( x \ll 1 \) obtained from calculating \( d_w = \partial (\log N_2)/\partial (\log R_2) \). The data includes walks with \( N_1 = 2000, 3000, 4000, 5000 \) constituting the fractal \( \text{RW} \) substrates, on which \( \langle R_2^3(N_1, N_2) \rangle \) was calculated by an exact enumeration procedure (Ben-Avraham and Havlin 1983, Magid et al 1984), up to \( N_2 = 2000 \). As seen from figure 1, the value \( d_w \) obtained is \( 3.5 \pm 0.1 \). Since \( d_f \) of the \( \text{RW} \) substrate is \( 2 \) we obtain \( \tilde{d} = 2d_i/d_w = 4/3.5 = 8/7 \). This result contradicts the suggested generalisation of the conjecture of AO that \( \tilde{d} \) should equal \( \frac{3}{2} \) (Meakin and Stanley 1983, Leyvraz and Stanley 1983). This result for \( \tilde{d} \) was also calculated directly, by analysing the probability of return to the origin from which it was found that \( \tilde{d}/2 = 0.57 \pm 0.02 \). The results for \( d_w \) and \( \tilde{d} \) in different dimensions are summarised in table 1.

![Figure 1](image-url)

**Figure 1.** Results for \( d_w \), the diffusion exponent of the second \( \text{RW} \) on the cluster generated by the first \( \text{RW} \) as a function of \( n = N_2 \) number of steps. The different symbols represent different values of \( N_i \): \( \bigcirc \) for \( N_i = 2000 \), \( \triangle \) for \( N_i = 3000 \), \( \bigcirc \) for \( N_i = 4000 \) and \( \square \) for \( N_i = 5000 \).
It is of interest to compare our results with a recent study (Banavar et al 1983) of the resistance between the end-to-end points of a RW. The resistance was found to be

\[ r(N_i, d) = N^{x_i(d)} \]

(11)

where \( N \) is the number of steps, \( x_i(d) = \frac{1}{2} \) for \( d = 1 \) and = \( \frac{d}{2} \) for \( d = 2, 3, 4 \) for the case when overlapping bonds are regarded as a single resistor. In order to relate our results to (11) we use the relation (Alexander and Orbach 1982)

\[ d_w = d_r + \bar{\xi} \]

(12)

where \( \bar{\xi} \) is the resistivity exponent defined by \( r = r^{\bar{\xi}} \) where \( r \) is the resistance between two bars of size \( R \) separated by a distance \( R \).

\[ r(N_i, d) = R^{2x_i(d)} = R^{\bar{\xi}(d)} \]

(13)

with \( \bar{\xi} = 1, 1.5, 2 \) for \( d = 1, 2, 3, 4 \) respectively. These results for \( \bar{\xi} \) (or \( x_i(d) \)) can be obtained by using (12). For \( d = 1 \) it is obvious that \( d_r = 1, d_w = 2, \bar{\xi} = 1 \). For \( d = 4 \) as mentioned earlier \( d_r = 2, d_w = 4 \), so that \( \bar{\xi} = 2 \). The case of \( d = 3 \) is not trivial. In this case \( d_r = 2 \) and our result \( d_w = 3.5 \) implies that \( \bar{\xi} = 1.5 \). The case of \( d = 2 \) is not consistent with (12) since we argue that the first walk occupies the entire space, \( d_w = 2, d_r = 2 \) thus \( \bar{\xi} \rightarrow 0 \) in contrast to \( \bar{\xi} = 1 \) obtained by Banavar et al (1983). This discrepancy may be explained by the fact that we study the case of \( N_i \rightarrow \infty \) and Banavar et al (1983) the case of finite \( N_i \). However, our general scaling assumptions (5) and (9) do not predict another exponent for this case (\( N_i = N_2 \)), it is rather represented as a crossover regime, (10).

To conclude, we have dealt with the problem of diffusion on a RW substrate. We solve exactly the one-dimensional case which suggests the form of the scaling function for all \( d \). For the non-trivial case of \( d = 3 \) we find \( d_w = 3.5 \) and \( \bar{d} = \frac{3}{2} \). These results are consistent with recent results on the resistivity for the RW substrate.

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