Modeling Morphology of Cities and Towns

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Predicting urban growth is important for practical reasons, and also for the challenge it presents to theoretical frameworks for cluster dynamics [1–3]. Recently, the model of diffusion limited aggregation (DLA) [4,5] has been applied to describe urban growth [1], and results in tree-like dendritic structures which have a core or "central business district" (CBD). The DLA model predicts that there exists only one large fractal cluster that is almost perfectly screened from incoming "development units" (people, capital, resources, etc), so that almost all the cluster growth occurs in the extreme peripheral tips. Here we propose and develop an alternative model to DLA that describes the morphology and the area distribution of systems of cities, as well as the scaling of the urban perimeter of individual cities. Our results agree both qualitatively and quantitatively with actual urban data. The resulting growth morphology can be understood in terms of the effects of interactions among the constituent units forming a urban region, and can be modeled using the correlated percolation model in the presence of a gradient.

In the model we now develop we take into account the following points:

(*i*) Urban data on the population density $\rho(r)$ of actual urban systems are known to conform to the relation [6] $\rho(r) = e^{-\lambda r}$, where r is the radial distance from the CBD situated at the core, and λ is the density gradient. Therefore, in our model the development units are positioned with an occupancy probability $p(r) \equiv \rho(r)$ that behaves in the same fashion as is known experimentally.

(*ii*) We also take into account the fact that in actual urban systems, the development units are not positioned at *random*. Rather, there exist *correlations* arising from the fact that when a development unit is located in a given place, the probability of adjacent development units increases naturally— i.e., each site is not independently occupied by a development unit, but is occupied with a probability that depends on the occupancy of the neighborhood.

In order to quantify these ideas, we consider the *correlated* percolation model [7–9]. In the limit where correlations are so small as to be negligible [10], a site at position r is occupied if the occupancy variable $u(r)$ is smaller than the occupation probability $p(r)$; the variables $u(r)$ are uncorrelated random numbers. To introduce correlation among the variables, we convolute the uncorrelated variables $u(r)$ with a suitable power law kernel [9], and define a new set of random variables $\eta(r)$ with long-range power-law correlations that decay as $r^{-\alpha}$, where $r \equiv |r|$. The assumption of power-law interactions is motivated by the fact that the "decision" for a development unit to be placed in a given location decays gradually with the distance from an occupied neighborhood. The correlation exponent α is the only parameter to be determined by empirical observations.

To discuss the morphology of a system of cities generated in the present model, we show in Fig. 1 our simulations of correlated urban systems for a fixed value of the density gradient λ, and for different degree of correlations. The correlations have the effect of agglomerating the units around a urban area. In the simulated systems the largest city is situated in the core, which is regarded as the attractive center of the city, and is surrounded by small towns. All towns are nearly compact near their centers and become less compact near their boundaries, in qualitative agreement with empirical data on actual large cities such as Berlin, Paris, London, etc. (see i.e. Refs. [1,11]).

The urban boundary of the largest city is defined to be the perimeter of the cluster connected to the CBD. Since $p(r)$ decreases as one moves away from the core, the probability that the largest cluster remains connected decreases with r. Hence r_f, the *mean* distance of the perimeter from the center, is determined by the value of r for which $p(r)$ equals the percolation threshold—i.e. $p(r_f) = p_c$, so $r_f = \lambda^{-1} \ln(1/p_c)$ [12,13].

The urban boundary in the model has the scaling properties of the external perimeter of a correlated percolation cluster in the presence of a gradient. The fractal dimension of the boundary, D_e is given by the scaling of the length of the boundary within a region of size ℓ, $L(\ell) \sim \ell^{D_e}$. We find $D_e \simeq 1.3$ for the uncorrelated case, and $D_e \simeq 1.4$ for strong correlations ($\alpha \to 0$). These values are consistent with actual urban data, for which values of D_e between 1.2 and 1.4 are measured [1]. Near the frontier and on length scales smaller than the width of the frontier, the largest cluster has fractal dimension $d_f \simeq 1.9$, as defined by the "mass-radius" relation [10], independent on the correlations.

So far, we have shown how the correlations between the occupancy probabilities can account for the irregular morphology of the towns in a urban system. As can be seen in Fig. 2*a*, the towns surrounding a large city like Berlin are characterized with a wide range of sizes. We are interested in the laws that quantify the town size distribution $N(A)$, where A is the area occupied by a given town or "mass" of agglomeration. To this end, we calculate the actual distribution of the areas of the urban settlements around Berlin and London. We first digitize the empirical data of Fig. 4.1 of Ref. [11] (which are shown in Fig. 2*a*), and Fig. 10.8 of Ref. [1]. Then, we count the number of towns that are covered by A sites, putting the result in logarithmicaly spaced bins, and averaging over the size of the bin. The results for the distributions $N(A)$ calculated in this way are shown in Fig. 3*a*, where we see that for both Berlin and London, $N(A)$ follows a power-law.

This new result of a power law area distribution of towns, $N(A)$, can be understood in the context of our model. Insight into this distribution can be developed by first noting that the small "towns" surrounding the largest city are all situated at distances r from the CBD such that $p(r) < p_c$ or $r > r_f$. Therefore, we find $N(A)$, the cumulative area distribution of clusters of area A, to be

$$N(A) \equiv \int_0^{p_c} n(A,p)\; dp \sim A^{-(\tau+1/d_f\nu)}. \qquad (1)$$

Here, $n(A,p) \sim A^{-\tau} g(A/A_0)$ is defined to be the average number of clusters containing A sites for a given p *at a fixed distance* r, and $\tau = 1 + 2/d_f$. Here, $A_0(r) \sim |p(r) - p_c|^{-d_f\nu}$ corresponds to the maximum typical area occupied by a "town" situated at a distance r from the CBD, while $g(A/A_0)$ is a scaling function that decays rapidly (exponentially) for $A > A_0$. The exponent $\nu = \nu(\alpha)$ is defined by $\xi(r) \sim |p(r) - p_c|^{-\nu}$, where $\xi(r)$ is the connectedness length that represents the mean linear extension of a cluster at a distance $r > r_f$ from the CBD.

In our numerical simulations we find a drastic increase of $\nu(\alpha)$ with the increase of the long-range correlations ($\alpha \to 0$) (Fig. 3*b*) [9]. The exponent $\nu(\alpha)$ affects the area distribution of the small clusters around the CBD (Fig. 3*c*), as specified by Eq.(1), and can be used to quantify the degree of interaction between the CBD and the small surrounding towns. For instance, for a strongly correlated system of cities characterized by small α, $\nu(\alpha)$ is large so that the area $A_0(r)$ and the linear extension $\xi(r)$ of the towns will be large even for towns situated away from the CBD. This effect is observed in the correlated systems of cities of Fig. 1.

In Fig. 3*a* we also plot the power-law for the area distribution predicted by Eq. (1) along with the real data for Berlin and London. We find that the slopes of the plots for both cities are consistent with the prediction (dashed line) for the case of highly correlated systems. These results quantify the qualitative agreement between the morphology of actual urban areas and the strongly correlated urban systems obtained in our simulations.

Next, we discuss a generalization of our static model to describe the dynamics of urban growth. Empirical studies [6] of the the population density profile of cities show a remarkable pattern of decentralization, which is quantified by the decrease of $\lambda(t)$ with time (see Fig. 4 and Table 4 in Ref. [14]). In the context of our model, this flattening pattern can be explained as follows. The model of percolation in a gradient can be related to a dynamical model of units (analog to the development units in actual cities) diffusing from a central seed or core [12,13]. In this dynamical system, the units are allowed to diffuse on a two-dimensional lattice by hopping to nearest-neighbor positions. The density of units at the core remains constant: whenever a unit diffuses away from the core, it is replaced by a new unit. Thus, the density of units is analog to the normalized population density of actual cities (see Fig. 4). A well-defined diffusion front, defined as the boundary of the cluster of units that is linked to the central core, evolves in time. The diffusion front corresponds to the urban boundary of the central city. The static properties of the diffusion front of this system were found to be the same as those predicted by the gradient percolation model [12,13]. Moreover, the dynamical model can explain the decrease of $\lambda(t)$ with time observed empirically. As the diffusion front situated around r_f moves away from the core, the city grows and the density gradient decreases since $\lambda(t) \propto 1/r_f$. Thus, the dynamics in the model are quantified by a decreasing $\lambda(t)$, as occurs in actual urban areas. These considerations are tested in Fig. 2*b*, which shows our dynamical urban simulations of a strongly interacting system of cities characterized by a correlation exponent $\alpha = 0.05$ for three different values of λ. Qualitative agreement is observed between the morphology of the cities and towns of the actual data of Fig. 2*a* and the simulations of Fig. 2*b*.

Before concluding, we discuss the implications of the present study. The striking coincidence between the settlement area distribution of Berlin (1920 and 1945) and London (1981) suggests that local planning policies (i.e., the Green Belt Policy applied in London in 1955 [1]) have a low impact on the distribution of towns. Although, a recent study shows that such policies may affect the fractal properties of the settlements [1]. The proposed model suggests that the area distribution is determined by the degree of interactions among development units, and that its scaling properties are independent of time. Thus, by extrapolating in time the behavior of λ, one might be able to obtain information on the future urban growth. It remains to be shown how universal our model is, in particular in relation to the presence of other factors such as topographical constraints, transportation routes, etc.

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FIG. 1. Simulations of urban systems for different degree of correlations. Here, the urban areas are red, and the external perimeter or urban boundary of the largest cluster connected to the CBD is light green. In all the figures, we fix the value of the density gradient to be $\lambda = 0.009$. (a) and (b) Two different examples of interactive systems of cities for correlation exponents $\alpha = 0.6$ and $\alpha = 1.4$, respectively. The development units are positioned with a probability that decays exponentially with the distance from the core. The units are located not randomly as in percolation, but rather in a correlated fashion depending of the neighboring occupied areas. The correlations are parametrized by the exponent α. The strongly correlated case corresponds to small α ($\alpha \to 0$). When $\alpha > d$, where d is the spatial dimension of the substrate lattice ($d = 2$ in our case), we recover the uncorrelated case. Notice the tendency to more compact clusters as we increase the degree of correlations ($\alpha \to 0$). (c) As a zeroth order approximation, one might imagine the morphology predicted in the extreme limit whereby development units are positioned at *random*, rather than in the correlated way of Figs. 1a and 1b. The results for this crude approximation of a non-interactive (uncorrelated) system of cities clearly display a drastically different morphology than found from data on real cities (such as shown in Fig. 2a). The non-interactive limit looks unrealistic in comparison with real cities, for the lack of interactions creates a urban area characterized by many small towns spread loosely around the core.

FIG. 2. Qualitative comparison between the actual urban data and the proposed model. (a) Three steps of the growth with time of Berlin and surrounding towns (from Ref. [11]). Data are shown for the years 1875, 1920, and 1945 (from top to bottom). The flattening of the density gradient is evident and corresponds to the decentralization of the urban area as the city grows. (b) Dynamical urban simulations of the proposed model. We fix the value of the correlation exponent to be $\alpha = 0.05$ (strongly correlated case), and choose the occupancy probability $p(r)$ to correspond to the data of Fig. 2a.

FIG. 3. (a) Log-log plot of the area distribution $N(A)$ of the actual towns around Berlin and London. In the case of Berlin, the data are shown for the years 1920 and 1945 (corresponding to the last two panels in Fig. 2a), while the data of London are for the year 1981. A power-law is observed for the area distributions of both urban systems. The dotted line shows the predictions of our model for the uncorrelated case (slope= 2.45), while the dashed line gives results for the strongly correlated case (slope= 2.06). Note that the area distributions for both cities agree much better with the strongly correlated case. The images have been digitized using an Apple Scanner, of resolution 150 dots per inch, and in this figure A denotes the number of sites covered by the development units for a given town. (b) Correlation length exponent $\nu(\alpha)$ as a function of the correlation exponent α. (c) Log-log plot of the area distribution $N(A)$ calculated for the present model for different degrees of correlation. From top to bottom, $\alpha = 0.2$, $\alpha = 0.8$, $\alpha = 1.4$, and uncorrelated case. The linear fits correspond to the predictions of Eq. (1) using the values of $\nu(\alpha)$ from Fig. 3b, and $d_f = 1.9$.

FIG. 4. Semi-log plot of the normalized density profile $\rho(r)$ ($\rho(0) = 1$) for the three different stages in the growth of Berlin shown in Fig. 2a, corresponding to the years 1875, 1920, and 1945 from bottom to top. Least square fits yield the results $\lambda \simeq 0.030$, $\lambda \simeq 0.012$, and $\lambda \simeq 0.009$, respectively, showing the decrease of λ with time. We used these density profiles in the dynamical simulations of Fig. 2b.