Modeling Morphology of Cities and Towns

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(April 29, 1999)

Predicting urban growth is important for practical reasons, and also for the challenge it presents to theoretical frameworks for cluster dynamics [1–3]. Recently, the model of diffusion limited aggregation (DLA) [4,5] has been applied to describe urban growth [1], and results in tree-like dendritic structures which have a core or “central business district” (CBD). The DLA model predicts that there exists only one large fractal cluster that is almost perfectly screened from incoming “development units” (people, capital, resources, etc.), so that almost all the cluster growth occurs in the extreme peripheral tips. Here we propose and develop an alternative model to DLA that describes the morphology and the area distribution of systems of cities, as well as the scaling of the urban perimeter of individual cities. Our results agree both qualitatively and quantitatively with actual urban data. The resulting growth morphology can be understood in terms of the effects of interactions among the constituent units forming a urban region, and can be modeled using the correlated percolation model in the presence of a gradient.

In the model we now develop we take into account the following points:

(i) Urban data on the population density \( \rho(r) \) of actual urban systems are known to conform to the relation [6] \( \rho(r) = e^{-\lambda r} \), where \( r \) is the radial distance from the CBD situated at the core, and \( \lambda \) is the density gradient. Therefore, in our model the development units are positioned with an occupancy probability \( p(r) \equiv \rho(r) \) that behaves in the same fashion as is known experimentally.

(ii) We also take into account the fact that in actual urban systems, the development units are not positioned at random. Rather, there exist correlations arising from the fact that when a development unit is located in a given place, the probability of adjacent development units increases naturally—i.e., each site is not independently occupied by a development unit, but is occupied with a probability that depends on the occupancy of the neighborhood.

In order to quantify these ideas, we consider the correlated percolation model [7–9]. In the limit where correlations are so small as to be negligible [10], a site at position \( r \) is occupied if the occupancy variable \( u(r) \) is smaller than the occupation probability \( p(r) \); the variables \( u(r) \) are uncorrelated random numbers. To introduce correlation among the variables, we convolute the uncorrelated variables \( u(r) \) with a suitable power law kernel [9], and define a new set of random variables \( u(r) \) with long-range power-law correlations that decay as \( r^{-\alpha} \), where \( \alpha \equiv 1/\alpha \).

The assumption of power-law interactions is motivated by the fact that the “decision” for a development unit to be placed in a given location decays gradually with the distance from an occupied neighborhood. The correlation exponent \( \alpha \) is the only parameter to be determined by empirical observations.

To discuss the morphology of a system of cities generated in the present model, we show in Fig. 1 our simulations of correlated urban systems for a fixed value of the density gradient \( \lambda \), and for different degree of correlations. The correlations have the effect of agglomerating the units around a urban area. In the simulated systems the largest city is situated in the core, which is regarded as the attractive center of the city, and is surrounded by small towns. All towns are nearly compact near their centers and become less compact near their boundaries, in qualitative agreement with empirical data on actual large cities such as Berlin, Paris, London, etc. (see i.e. Refs. [1,11]).

The urban boundary of the largest city is defined to be the perimeter of the cluster connected to the CBD. Since \( p(r) \) decreases as one moves away from the core, the probability that the largest cluster remains connected decreases with \( r \). Hence \( r_f \), the mean distance of the perimeter from the center, is determined by the value of \( r \) for which \( p(r) \) equals the percolation threshold—i.e. \( p(r_f) = p_c \), so \( r_f = \lambda^{-1} \ln(1/p_c) \) [12,13].

The urban boundary in the model has the scaling properties of the external perimeter of a correlated percolation cluster in the presence of a gradient. The fractal dimension of the boundary, \( D_c \), is given by the scaling of the length of the boundary within a region of size \( \ell \), \( L(\ell) \sim \ell^{D_c} \). We find \( D_c \approx 1.3 \) for the uncorrelated case, and \( D_c \approx 1.4 \) for strong correlations (\( \alpha \to 0 \)). These values are consistent with actual urban data, for which values of \( D_c \) between 1.2 and 1.4 are measured [1]. Near the frontier and on length scales smaller than the width of the frontier, the largest cluster has fractal dimension \( d_f \approx 1.9 \), as defined by the “mass-radius” relation [10], independent on the correlations.

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So far, we have shown how the correlations between
the occupancy probabilities can account for the irregular
morphology of the towns in a urban system. As can be
seen in Fig. 2a, the towns surrounding a large city like
Berlin are characterized with a wide range of sizes. We
are interested in the laws that quantify the town size dis-
tribution \( N(A) \), where \( A \) is the area occupied by a given
town or “mass” of agglomeration. To this end, we calcu-
late the actual distribution of the areas of the urban set-
tlements around Berlin and London. We first digitize the
empirical data of Fig. 4.1 of Ref. [11] (which are shown
in Fig. 2a), and Fig. 10.8 of Ref. [1]. Then, we count the
number of towns that are covered by \( A \) sites, putting the
result in logarithmically spaced bins, and averaging
over the size of the bin. The results for the distributions
\( N(A) \) calculated in this way are shown in Fig. 3a, where
we see that for both Berlin and London, \( N(A) \) follows a
power-law.

This new result of a power law distribution of
towns, \( N(A) \), can be understood in the context of our
model. Insight into this distribution can be developed by
first noting that the small “towns” surrounding the
largest city are all situated at distances \( r \) from the CBD
such that \( p(r) < p_c \) or \( r > r_f \). Therefore, we find \( N(A) \),
cumulative area distribution of clusters of area \( A \), to be

\[
N(A) \equiv \int_0^\infty n(A, p) \, dp \sim A^{-(\tau + 1/d_f \nu)}.
\]

Here, \( n(A, p) \sim A^{-\gamma} g(A/A_0) \) is defined to be the aver-
age number of clusters containing \( A \) sites for a given \( p
\)
at a fixed distance \( r \), and \( \tau = 1 + 2/d_f \nu \). Here, \( A_0 = A_0(r) \sim \)
\( [p(r) - p_c]^{-d_f \nu} \) corresponds to the maximum typical area
occupied by a “town” situated at a distance \( r \) from the
CBD, while \( g(A/A_0) \) is a scaling function that
decays rapidly (exponentially) for \( A > A_0 \). The exponent
\( \nu = \nu(\alpha) \) is defined by \( \xi(r) \sim \) \( [p(r) - p_c]^{-\nu} \), where \( \xi(r) \) is the
connected length that represents the mean linear
extension of a cluster at a distance \( r > r_f \) from the
CBD.

In our numerical simulations we find a drastic in-
crease of \( \nu(\alpha) \) with the increase of the long-range corre-
lations \( (\alpha \to 0) \) (Fig. 3b) [9]. The exponent \( \nu(\alpha) \) affects the area
distribution of the small clusters around the CBD (Fig.
3c), as specified by Eq. (1), and can be used to quanti-
tify the degree of interaction between the CBD and the small
surrounding towns. For instance, for a strongly corre-
slated system of cities characterized by small \( \alpha \), \( \nu(\alpha) \) is large so
that the area \( A_0(r) \) and the linear extension \( \xi(r) \) of the
towns will be large even for towns situated away from the
CBD. This effect is observed in the correlated systems of
cities of Fig. 1.

In Fig. 3a we also plot the power-law for the area dis-
tribution predicted by Eq. (1) along with the real data
for Berlin and London. We find that the slopes of
the plots for both cities are consistent with the prediction
(dashed line) for the case of highly correlated systems.
These results quantify the qualitative agreement between

the morphology of actual urban areas and the strongly
correlated urban systems obtained in our simulations.

Next, we discuss a generalization of our static model
to describe the dynamics of urban growth. Empirical stud-
ies [6] of the the population density profile of cities show
a remarkable pattern of decentralization, which is quan-
tified by the decrease of \( \lambda(t) \) with time (see Fig. 4 and
Table 4 in Ref. [14]). In the context of our model, this
flattening pattern can be explained as follows. The model
of percolation in a gradient can be related to a dynamical
model of units (analog to the development units in actual
cities) diffusing from a central seed or core [12, 13]. In this
dynamical system, the units are allowed to diffuse on a
two-dimensional lattice by hopping to nearest-neighbor
positions. The density of units at the core remains con-
stant; whenever a unit diffuses away from the core, it is
replaced by a new unit. Thus, the density of units is ana-
log to the normalized population density of actual cities
(see Fig. 4). A well-defined diffusion front, defined as the
boundary of the cluster of units that is linked to the cen-
tral core, evolves in time. The diffusion front corresponds
to the urban boundary of the central city. The static
properties of the diffusion front of this system were found
to be the same as those predicted by the gradient per-
colation model [12, 13]. Moreover, the dynamical model
can explain the decrease of \( \lambda(t) \) with time observed em-
pirically. As the diffusion front situated around \( r_f \) moves
away from the core, the city grows and the density gra-
dient decreases since \( \lambda(t) \propto 1/r_f \). Thus, the dynamics in
the model are quantified by a decreasing \( \lambda(t) \), as occurs
in actual urban areas. These considerations are tested in
Fig. 2b, which shows our dynamical urban simulations of
a strongly interacting system of cities characterized by a
correlation exponent \( \alpha \equiv 0.05 \) for three different values of
\( \lambda \). Qualitative agreement is observed between the mor-
phology of the cities and towns of the actual data of Fig.
2a and the simulations of Fig. 2b.

Before concluding, we discuss the implications of the
present study. The striking coincidence between the set-
tlement area distribution of Berlin (1920 and 1945) and
London (1981) suggests that local planning policies (i.e.,
the Green Belt Policy applied in London in 1955 [1]) have
a low impact on the distribution of towns. Although, a recen
t study shows that such policies may affect the frac-
tal properties of the settlements [1]. The proposed model
suggests that the area distribution is determined by the
degree of interactions among development units, and that
its scaling properties are independent of time. Thus, by
extrapolating in time the behavior of \( \lambda \), one might be
able to obtain information on the future urban growth.
It remains to be shown how universal our model is, in par-
sicular in relation to the presence of other factors such as
topographical constraints, transportation routes, etc.
FIG. 1. Simulations of urban systems for different degree of correlations. Here, the urban areas are red, and the external perimeter or urban boundary of the largest cluster connected to the CBD is light green. In all the figures, we fix the value of the density gradient to be \( \lambda = 0.009 \). (a) and (b) Two different examples of interactive systems of cities for correlation exponents \( \alpha = 0.6 \) and \( \alpha = 1.4 \), respectively. The development units are positioned with a probability that decays exponentially with the distance from the core. The units are located not randomly as in percolation, but rather in a correlated fashion depending of the neighboring occupied areas. The correlations are parametrized by the exponent \( \alpha \). The strongly correlated case corresponds to small \( \alpha (\alpha \to 0) \). When \( \alpha > d \), where \( d \) is the spatial dimension of the substate lattice \( (d = 2 \) in our case), we recover the uncorrelated case. Notice the tendency to more compact clusters as we increase the degree of correlations \( (\alpha \to 0) \). (c) As a zeroth order approximation, one might imagine the morphology predicted in the extreme limit whereby development units are positioned at random, rather than in the correlated way of Figs. 1a and 1b. The results for this crude approximation of a non-interactive (uncorrelated) system of cities clearly display a drastically different morphology than found from data on real cities (such as shown in Fig. 2a). The non-interactive limit looks unrealistic in comparison with real cities, for the lack of interactions creates a urban area characterized by many small towns spread loosely around the core.

FIG. 2. Qualitative comparison between the actual urban data and the proposed model. (a) Three steps of the growth with time of Berlin and surrounding towns (from Ref. [11]). Data are shown for the years 1875, 1920, and 1945 (from top to bottom). The flattening of the density gradient is evident and corresponds to the decentralization of the urban area as the city grows. (b) Dynamical urban simulations of the proposed model. We fix the value of the correlation exponent to be \( \alpha = 0.05 \) (strongly correlated case), and choose the occupancy probability \( p(r) \) to correspond to the data of Fig. 2a.

FIG. 3. (a) Log-log plot of the area distribution \( N(A) \) of the actual towns around Berlin and London. In the case of Berlin, the data are shown for the years 1930 and 1945 (corresponding to the last two panels in Fig. 2a), while the data of London are for the year 1981. A power-law is observed for the area distributions of both urban systems. The dotted line shows the predictions of our model for the uncorrelated case \( (\text{slope} = 2.45) \), while the dashed line gives results for the strongly correlated case \( (\text{slope} = 2.60) \). Note that the area distributions for both cities agree much better with the strongly correlated case. The images have been digitized using an Apple Scanner, of resolution 150 dots per inch, and in this figure A denotes the number of sites covered by the development units for a given town. (b) Correlation length exponent \( \nu(\alpha) \) as a function of the correlation exponent \( \alpha \). (c) Log-log plot of the area distribution \( N(A) \) calculated for the present model for different degrees of correlation. From top to bottom, \( \alpha = 0.2, \alpha = 0.8, \alpha = 1.4, \) and uncorrelated case. The linear fits correspond to the predictions of Eq. (1) using the values of \( \nu(\alpha) \) from Fig. 3b, and \( d_f = 1.9 \).

FIG. 4. Semi-log plot of the normalized density profile \( \rho(r) \) \( (\rho(0) = 1) \) for the three different stages in the growth of Berlin shown in Fig. 2a, corresponding to the years 1875, 1920, and 1945 from bottom to top. Least square fits yield the results \( \lambda \approx 0.630, \lambda \approx 0.012, \) and \( \lambda \approx 0.009 \), respectively, showing the decrease of \( \lambda \) with time. We used these density profiles in the dynamical simulations of Fig. 2b.