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Structural and dynamical properties of random walk clusters

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Abstract. We study the structural and dynamical properties of the clusters generated by a nearest-neighbour random walk embedded in a d-dimensional space. We have focused on the non-trivial case in which the cluster is generated in $d = 3$. The structure of this cluster is characterised by loops for all length scales on the one hand and by the fact that deadends are negligible (upon scaling) on the other hand. The cluster is very dilute and is characterised by fractal dimension $d_f = 2$ and chemical dimension $d_l = 1.29 \pm 0.04$. From these results it follows that $\tilde{\nu} \equiv d_l/d_f \simeq \frac{2}{3}$, which is consistent with the formula $\tilde{\nu} = 2/d$ $(2 \leq d \leq 4)$, obtained using a Flory-type argument. The dynamical diffusion exponents d_w and d_w^l were calculated using the exact enumeration method and found to be $d_w = 3.45 \pm 0.10$ and $d_w^l = 2.28 \pm 0.05$. Our results suggest that the effect of loops is small but not negligible. We also calculated the fracton dimensionality of the cluster and obtained $d_s = 1.14 \pm 0.02$. A scaling function is presented for the end-to-end mean square displacement of a random walk performed on a random walk cluster. This scaling function is supported by our numerical results.

1. Introduction

The structure of random aggregates is currently a subject of intensive study [1-8]. The main reason for the recent interest in growth models is their relevance to experiment [9]. Another reason for this interest is that growth models exhibit a new type of dynamical scaling, which is related to the geometrical properties of the aggregate. The structural and dynamical properties of aggregation models, such as diffusion-limited aggregation (DLA), cluster-cluster aggregation and cluster growth models, have been extensively studied. However, the structure of the cluster generated by a random walk has received little consideration [10-12]. The main properties of interest are the fractal dimension d_f, transport properties characterised by diffusion and resistivity exponents (d_w and $\tilde{\zeta}$, respectively) and the fracton dimensionality [1] $d_s = 2d_f/d_w$.

In this paper we study the fundamental problem of the structure and properties of the fractal generated by a nearest-neighbour random walk (RW) of N steps embedded in d-dimensional space. The properties of RW have been studied for many years [13], both for their intrinsic mathematical interest and for their application to a variety of physical problems in solid state physics and in polymer physics. Although many properties of the RW model are now understood, a complete study has never been carried out of the structure of the RW cluster, i.e. the random aggregate generated by the RW. A principal application of this model is the transport properties in porous sedimentary rocks of low porosity [14]. It has been observed that such materials can retain a connected rod space down to extremely low values of porosity. This fact is inconsistent with a common approach to microscopically disordered random media

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in which the pore space is modelled as a percolation network and becomes disconnected at much larger porosity. The pore space is presumably correlated and a RW cluster was suggested [10] as a model which guarantees its connectedness.

This paper is organised as follows. In § 2, we introduce and describe the two possible models of the RW cluster. In § 3, we discuss its fractal dimension and its relation to the fractal dimension of the RW that generates the cluster. The structure of the cluster in the chemical metric is studied both analytically and numerically in § 4. We discuss the transport properties of the RW cluster in § 5. In § 6 we derive a scaling form for the end-to-end mean square displacement of an N_2-step RW performed on a cluster generated by an N_1-step RW. In § 7, we present a numerical calculation of the diffusion exponents, d_w and d_w^l and the fracton dimensionality d_s in $d=3$. We also present numerical results which support our scaling form suggested in § 6. A discussion of the results and remaining open questions are presented in § 8.

2. The random walk cluster models

We have studied cluster aggregates generated by the following growth method. A random walker starts at the origin of a d-dimensional hypercubic lattice. Each bond visited by the RW is considered an element of the cluster. There are two possibilities to define the mass and the conductivity of each bond in the cluster. (*a*) The mass and the conductivity of each bond is proportional to the number of times the RW has passed through it. (*b*) The mass and the conductivity of each bond is unity, independent of the number of times it has been visited.

The two RW cluster models are in the same universality class when generated in space with $d \geq 4$. This is due to the fact that, for a RW in $d \geq 4$, loops can be neglected [15] and both RW models are in the same universality class as SAW. In this case, the mass of the cluster is proportional to the number of steps of the RW (or SAW) which generate the cluster. For $d < 4$, loops in RW cannot be neglected, and thus a RW cluster is different from a SAW cluster which, by definition, has no loops.

The above two definitions for the RW cluster raise the following questions. Is there a critical dimension above which the two models belong to the same universality class below $d=4$? What are the physical properties of the two models above the critical dimension and what are the differences between the two models below it? These questions will be discussed in the following sections.

3. The fractal dimension of the cluster

The fractal dimension d_f of a RW cluster is defined by the relation

$$\langle R^2(N)\rangle^{d_f/2} \sim M(N) \tag{3.1}$$

where $R(N)$ is the linear size of the cluster generated by a RW performing N steps and $M(N)$ is the mass of the cluster defined in § 2. The exponent d_f is a measure of the compactness of the cluster. Another exponent characterising a RW is the diffusion exponent d_w defined by the relation

$$\langle R^2(N)\rangle^{d_w/2} \sim N. \tag{3.2}$$

The exponent d_w is a measure of the winding of the random walk. From (3.1) and (3.2) it is easy to see that both definitions coincide when the mass $M(N)$ scales as N.

A simple example in which this condition is fulfilled is the cluster generated by a self-avoiding random walk (SAW).

In model (*a*), the mass of each cluster element is proportional to the number of times it has been visited by the RW and thus the mass of the cluster $M(N)$ scales as N from which it follows that both definitions of d_w and d_f coincide for all d. This is not the case for model (*b*), since the mass of the cluster in model (*b*) is proportional to the number of distinct sites $S(N)$ visited by a RW which scales as

$$M(N) \sim S(N) \sim \begin{cases} N^{d/2} & d < 2 \\ N/\ln N & d = 2 \\ N & d > 2. \end{cases} \tag{3.3}$$

From this functional form we conclude that, for clusters generated in $d \geq 2$, the two definitions of the fractal dimension coincide. However, d_w and d_f are different for clusters generated in $d < 2$. From the above discussion, it follows that $d = 2$ is a critical dimension above which the two models belong to the same universality class. Note that when the RW cluster is generated on a *fractal* substrate with fracton dimension d_s, the Euclidean dimension d appearing in (3.3) and in the above conclusions should be changed to d_s [1]. In this case, the critical dimension is $d_s = 2$.

The fractal dimension of the cluster in model (*a*) is $d_f = 2$, independent of d, and for a RW cluster generated on a fractal substrate, d_f is equal to the fractal dimension of the diffusion on the substrate. In model (*b*) one must distinguish between two cases. When the RW cluster is generated in space with $d \leq 2$, it is space filling, from which it follows that its fractal dimension is equal to the dimension of the space. (In the case where the RW cluster is generated on a fractal substrate with $d_s < 2$, then d_f is equal to the fractal dimension of the substrate.) When the cluster is generated in space with $d > 2$, its fractal dimension is $d_f = 2$, independent of d, since $d_w = 2$ for RW in Euclidean space. When calculating d_f in numerical simulations we expect to observe correction to the asymptotic value $d_f = 2$. This results from the correction to the scaling form of $S(N)$ for finite N [13]. In $d = 2$, $S(N)$ scales as $N/\log(N)$ from which it follows that $d_f = 2 - (2/N) \ln N$. In $d = 3$, $S(N)$ scales as $N + a\sqrt{N}$ and thus $d_f = 2 - a/(\sqrt{N} \log N)$, and in $d = 4$, $S(N)$ scales as $N + a \log(N)$, from which it follows that $d_f = 2 - a/N$. As can be seen, the correction becomes less important as we increase the dimension of the space in which the cluster is generated.

4. The chemical dimension

Two metrics have been found useful for the characterisation of distance on a fractal [6]. One of these is obviously the geometric distance r and the second is the chemical distance l defined as the shortest distance between two points as measured along the structure. In analogy to the fractal dimension which characterises the compactness of the cluster in the Euclidean metric, we can define the chemical dimension d_l (or graph dimension) which characterises the compactness of the cluster in the chemical metric:

$$\overline{M(l)} \sim l^{d_l} \tag{4.1}$$

where $M(l)$ is the average mass of a subcluster which consists of l chemical shells. A chemical shell consists of all sites that are at a chemical distance l from an arbitrary

site defined as the origin. The two metrics are connected by the exponent $\tilde{\nu}$ which is defined by the relation

$$\overline{R^2(l)} \sim l^{2\tilde{\nu}} \tag{4.2}$$

where $\overline{R^2(l)}$ is the mean square radius of the lth chemical shell. The physical interpretation of this definition is that the exponent $\tilde{\nu}$ is a measure of the winding of the shortest path connecting pairs of sites in the cluster. Using the definitions of the fractal dimension, $M \sim R^{d_f}$ and the graph dimension $M \sim l^{d_l}$ we conclude that the exponent $\tilde{\nu}$ is

$$\tilde{\nu} = d_l / d_f. \tag{4.3}$$

The exponent $\tilde{\nu}$ can be shown to be bounded by

$$\left(\frac{3}{2+d} \simeq\right) \nu_{\mathrm{SAW}} \leqslant \tilde{\nu} \leqslant 1. \tag{4.4}$$

The upper bound is due to the fact that the minimum value of l is R (as in Euclidean space). The lower bound is the exponent ν of SAW ($\langle R^2(N)\rangle = aN^{2\nu}$), where in the parenthesis of (4.4) we give the Flory approximate value of ν for SAW [16]. This bound arises from the inequality $N \geqslant l$ (where N is the number of steps the random walk has performed between visiting the two sites). In SAW clusters $N = l$ and the two exponents $\tilde{\nu}$ and ν_{SAW} coincide. Since loops are allowed in RW clusters, $N \geqslant l$ and it thus follows that $\tilde{\nu} \geqslant \nu_{\mathrm{SAW}}$.

In model (a) the mass of the cluster is proportional to the number of steps, $M(l) \propto l^{d_l} \propto N$ and $R^2 \propto N \propto l^{2\tilde{\nu}}$, from which it follows that

$$2\tilde{\nu} = d_l \qquad \text{for all } d. \tag{4.5}$$

This relation is valid also for model (b) for clusters generated in $d \geqslant 2$.

The values for $\tilde{\nu}$ and d_l in $d = 1$, 2 and 4 dimensions can be evaluated (see table 1) by the following simple arguments. The two models differ only for $d = 1$. In model (a), $d_f = 2$ and, since $R \equiv l$, it follows that $\tilde{\nu} = 1$ and $d_l = 2$. For model (b), $d_f = 1$ and, since $R = l$, it follows that $\tilde{\nu} = 1$ and $d_l = 1$. In $d = 2$ for $N \to \infty$, the cluster spans the whole space uniformly. Thus, it is characterised by $d_f = 2$ and $R \equiv l$, from which it follows that $\tilde{\nu} = 1$ and $d_l = 2$ for both models. For $d \geqslant 4$, loops can be neglected upon scaling [16], i.e. $N \propto l$, from which it follows that $\tilde{\nu} = \frac{1}{2}$ and $d_l = 1$. These results are summarised in table 1. When the cluster is generated in $d = 3$, the exponents d_l and $\tilde{\nu}$ cannot be evaluated using the above simple arguments. This occurs because, on the one hand, the cluster is not space filling (as in $d = 2$) and, on the other hand, the loops in the cluster cannot be neglected (as in $d \geqslant 4$). Thus, we used numerical simulation to calculate d_l and $\tilde{\nu}$ for clusters generated in model (b) in $d = 3$. We calculated $M(l)$ and $R^2(l)$ averaged typically over 1000 clusters (see figures 1 and 2). It is seen from

Table 1. Summary of exponents characterising RW clusters.

d	d_f	d_l	$\tilde{\nu}$	$d_w^{(2)}$	$d_s^{(2)}$
1	$1^b/2^a$	$1^b/2^a$	1	2	$1^b/2^a$
2	2	2	1	2.5 ± 0.05	1.6
3	2	1.29 ± 0.04	0.68 ± 0.02	3.45 ± 0.10	1.14 ± 0.02
4	2	1	$\frac{1}{2}$	4	1

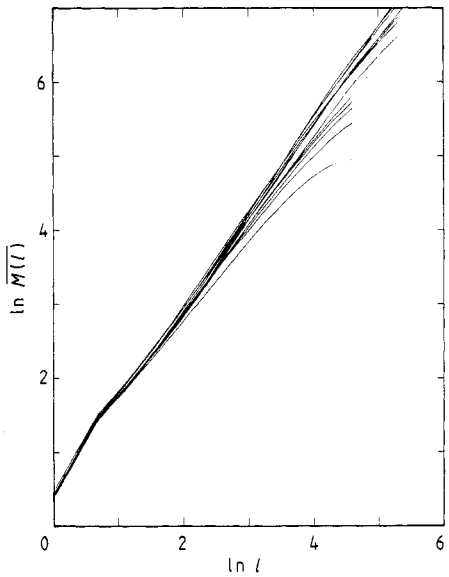

Figure 1. Plot of the average $M(l)$ of the cluster generated by a RW in a three-dimensional cubic lattice as a function of l on a double logarithmic scale. The clusters were generated by a RW of 200, 400, 600, 800, 1000, 2000, 3000, 4000, 5000, 10 000 and 15 000 steps. From the slope of these graphs, we obtain $d_l = 1.29 \pm 0.04$.

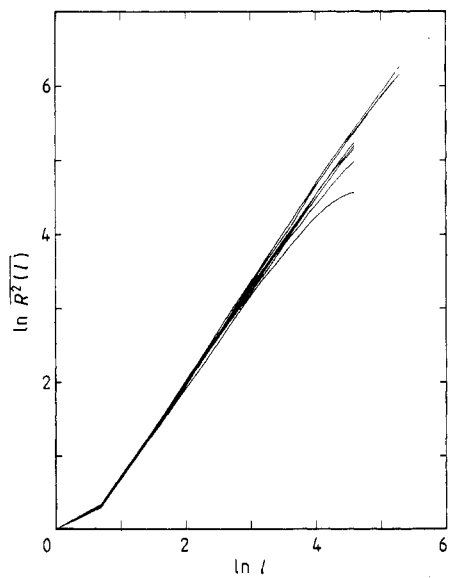

Figure 2. Plot of the mean square radius $\overline{R^2(l)}$ of the cluster generated by a RW in a three-dimensional cubic lattice as a function of l on a double logarithmic scale. The clusters were generated by RW in the lengths mentioned in figure 1. From the slope of these graphs, we obtain $2\tilde{\nu} = 1.36 \pm 0.04$.

these figures that, by increasing the number of steps of the generating random walker, the slope of the graph becomes larger and converges to an asymptotic value. From the asymptotic slopes we obtained $d_l = 1.29 \pm 0.04$ and $2\tilde{\nu} = 1.36 \pm 0.04$. The exponent $\tilde{\nu}$ was also calculated from dynamical exponents. This method yields $\tilde{\nu} \simeq \frac{2}{3}$ which is very close to the above results and will be discussed later in § 7.

The results summarised in table 1 suggest the following conjecture:

$$\tilde{\nu} = 2/d \tag{4.6}$$

implying that the lower critical dimension for this problem is $d_c^L = 2$ and the upper critical dimension is $d_c^U = 4$. To support (4.6), we present a Flory-type argument based on the following free energy approximation:

$$F \sim \frac{R^2}{l} + \frac{R^2}{R^d} l. \tag{4.7}$$

Equation (4.7) contains two terms: the first is the usual term representing the entropy whereas the second term represents the interaction energy. The interaction energy is assumed to be proportional to the number of crossings in the path of length l, which scales as l times R^2/R^d which is the probability that a site is occupied. Minimising the free energy we obtain $R^d \propto l^2$, from which follows (4.6).

5. Transport properties of RW clusters

When studying the transport properties, one has to distinguish between two regimes: the cluster generated in space with d_s below 2 and with d_s above 2. This distinction results from the fact that $S(N)$ scales as [1]

$$S(N) \sim \begin{cases} N & d_s^{(1)} > 2 \\ N^{d_s^{(1)}/2} & d_s^{(1)} \leqslant 2. \end{cases} \tag{5.1}$$

The index (1) represents exponents characterising the space in which the cluster was generated and the index (2) represents the exponents which characterise the generated RW cluster. Since $S(N) \propto R(N)^{d_f^{(2)}}$, it follows that

$$R(N) \sim \begin{cases} N^{1/d_f^{(2)}} & d_s^{(1)} > 2 \\ N^{d_s^{(1)}/2d_f^{(2)}} & d_s^{(1)} \leqslant 2. \end{cases} \tag{5.2}$$

The exponent $\tilde{\zeta}$ characterising the cluster resistivity is defined by the relation

$$\rho(N) \sim R(N)^{\tilde{\zeta}} \tag{5.3}$$

where $\rho(N)$ is the average resistivity of a cluster generated by a N-step RW. Using the Einstein relation between diffusion and transport exponents [17]

$$\tilde{\zeta} = d_w^{(2)} - d_f^{(2)} \tag{5.4}$$

where $d_w^{(2)}$ is the fractal dimension of the diffusion on the cluster (RW on a RW), we obtain

$$\rho(N) \sim S(N)^{2/d_s^{(2)}-1} \sim \begin{cases} N^{2/d_s^{(2)}-1} & d_s^{(1)} > 2 \\ N^{\frac{1}{2}d_s^{(1)}(2/d_s^{(2)}-1)} & d_s^{(1)} \leqslant 2. \end{cases} \tag{5.5}$$

In the case $d_s^{(1)} < 2$ we have to distinguish between the two models. In model (b) the cluster generated is space filling, i.e. $d_f^{(2)} = d_f^{(1)}$ and $d_w^{(2)} = d_w^{(1)}$, from which it follows that $d_s^{(2)} = d_s^{(1)}$ and

$$\rho(N) \sim N^{1-d_s^{(1)}/2} \qquad d_s^{(1)} < 2. \tag{5.6a}$$

In the first model (model (*a*)), the fractal dimension of the cluster is equal by definition to the fractal dimension of the diffusion on the space ($d_f^{(2)} = d_w^{(1)}$). Since the first RW is homogeneously space filling, the second RW visits essentially the same space as the first RW, i.e. $d_w^{(2)} = d_w^{(1)}$. Thus, $d_s^{(2)} = 2$ and

$$\rho(N) \sim N^0 \sim \text{constant} \qquad d_s^{(1)} < 2. \tag{5.6b}$$

For example, when the cluster is generated in $d = 1$, then in model (*a*), $\tilde{\zeta} = 0$† whereas $\tilde{\zeta} = 1$ in model (*b*). When the cluster is generated by a RW of infinite length ($N_1 \to \infty$) in $d = 2$, there is no distinction between the two models since $S(N) \propto N$ and $\tilde{\zeta} = 0$ in both models. Note that, as will be shown in § 7, when the cluster is generated by a random walk of finite length, effectively $d_w^{(2)} > 2$ from which follows $\tilde{\zeta} > 0$. When the cluster is generated in space with $d_s^{(1)} > 4$, then using the well known feature that the loops in such a cluster can be neglected upon scaling, we obtain $d_w^{(2)} = 4$ and $\tilde{\zeta} = 2$‡. For clusters generated in space with $2 < d < 4$, the value of $\tilde{\zeta}$ was calculated by Banavar *et al* [10]

$$\tilde{\zeta} = \tfrac{1}{2}d \qquad d_s^{(2)} = \frac{8}{4+d}. \tag{5.7}$$

In § 7 we present a numerical calculation for clusters generated in $d = 3$ from which we obtain $d_w^{(2)} = 3.45$ and $d_s^{(2)} = 1.14$ which is in good agreement with the theoretical results of (5.7). Note also that our finding that $d_w^{(2)} = 3.45 = 2 + \tilde{\zeta}$ is in good agreement with the recent $\varepsilon = 4 - d$ expansion result [19] $\tilde{\zeta} = 2(1 + \frac{1}{4}\varepsilon - \frac{1}{16}\varepsilon^2)$.

6. RW on RW scaling function

In this section, we study the scaling form of the function $\langle R_2^2(N_1, N_2)\rangle$, defined as the end-to-end mean square distance of an N_2-step RW performed on a cluster generated by an N_1-step RW. In $d = 1$, this function can be evaluated analytically by calculating the integral [11]

$$\langle R_2^2(N_1, N_2)\rangle = \int_0^\infty \langle R^2(x, N_2)\rangle g(x, N_1)\, dx \tag{6.1}$$

where $g(x, N_1)$ is the probability density for the span x of a RW cluster of N_1 steps and $\langle R^2(x, N_2)\rangle$ is the mean square end-to-end distance of an N_2-step RW performed on a segment of length x. By calculating the integral (6.1), we find that $\langle R_2^2(N_1, N_2)\rangle$ is of the form

$$\langle R_2^2(N_1, N_2)\rangle \simeq N_1^a N_2^b f(N_2^c/N_1) \tag{6.2}$$

with $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = 1$. We assume that this scaling form applies for any dimensionality d with suitable values of a, b and c. In fact, the limits of $f(x = N_2^c/N_1)$ for $x \to 0$ and $x \to \infty$ are easily obtained from simple arguments. For $x \to \infty$, the number of steps taken by the second RW is much larger than the number taken by the first. Thus, the span of the first RW limits $\langle R_2^2\rangle$ to be proportional to N_1 and independent of N_2. For

† For this case (*a*) it was found recently in [18] that $\rho(N) \sim \frac{1}{2}\ln N$ for $d = 1$.

‡ In the case of random walk on a Cayley tree which represents high dimension, an exact solution was found recently by Harris (unpublished).

$x \to 0$, the RW is, with overwhelming probability, far from the edges of the cluster generated by the first RW. Therefore, $\langle R_2^2 \rangle \sim N_2^{2/d_w}$, independent of N_1, leading to

$$f(x) = \begin{cases} x^a & x \to 0 \\ x^{-b/c} & x \to \infty. \end{cases} \tag{6.3}$$

Substituting (6.3) into (6.2) yields

$$\langle R_2^2(N_1, N_2) \rangle \sim \begin{cases} N_2^{b+ca} & N_1 \to \infty, \ N_2 \text{ finite} \\ N_1^{(b+ca)/c} & N_2 \to \infty, \ N_1 \text{ finite.} \end{cases} \tag{6.4}$$

Our earlier remarks therefore imply the relation

$$c = 2/d_w \qquad \tfrac{1}{2}bd_w + a = 1. \tag{6.5}$$

Introducing these identities into (6.2) leads to

$$\frac{\langle R_2^2(N_1, N_2) \rangle}{N_1} \simeq \left(\frac{N_2^{2/d_w}}{N_1}\right)^{bd_w/2} f\left(\frac{N_2^{2/d_w}}{N_1}\right) = g\left(\frac{N_2^{2/d_w}}{N_1}\right) \tag{6.6}$$

where

$$g(x) \sim \begin{cases} x & x \ll 1 \\ \text{constant} & x \gg 1. \end{cases} \tag{6.7}$$

Thus, we see that only a single exponent $2/d_w$ suffices to describe limiting properties of $\langle R_2^2(N_1, N_2) \rangle$. Similar arguments yield that the function $\langle l_2(N_1, N_2) \rangle$, the mean square chemical distance of an N_2-step RW performed on a cluster generated by an N_1-step RW, can be described by $2/d_w^l$ as

$$\frac{\langle l_2(N_1, N_2) \rangle}{N_1} \sim h\left(\frac{N_2^{1/d_w^l}}{N_1}\right) \tag{6.8}$$

where

$$h(x) \sim \begin{cases} x & x \ll 1 \\ \text{constant} & x \gg 1. \end{cases} \tag{6.9}$$

In the next section, we will present numerical results which support the scaling forms given in (6.6)-(6.9).

7. Dynamical properties

In this section, we present numerical calculations of the dynamical exponents $d_w^{(2)}$ and $d_w^{l(2)}$ characterising the cluster generated by a RW in $d = 3$. We also calculate the fracton dimension of the cluster $d_s^{(2)}$ both directly by measuring the probability to return to the origin and by using the relation $d_s^{(2)} = 2d_f^{(2)}/d_w^{(2)} = 2d_l^{(2)}/d_w^{l(2)}$. The numerical calculations were performed using simulation techniques in which we first generate a RW cluster and then measure the mean square displacement in the geometrical and chemical metrics to obtain the dynamical exponents.

The clusters were generated by performing a RW on a cubic lattice. We use a technique in which we store only the coordinates of the RW without storing the entire lattice. This technique enables us to perform very long RW. However, the price we pay is that the time needed to perform a RW of N_1 steps is proportional to N_1^2 instead of N_1. On each cluster generated, we perform a second RW using the exact enumeration algorithm [20], and measure the end-to-end displacement of the second RW from its origin after N_2 steps. The results were averaged over an ensemble of typically 500 clusters, each consisting of N_1 steps.

The exponent $d_w^{(2)}$ characterising the diffusion on the RW cluster in the geometric metric was calculated from the slope of the graphs in figure 3. In these figures, we plotted on a double logarithmic scale $\langle R^2(N_2)\rangle$ as a function of N_2 and $\langle R^4(N_2)\rangle$ as a function of N_2. From the slopes, we obtained $d_w^{(2)} = 3.45 \pm 0.10$. Similarly, we calculated the exponent $d_w^{l(2)}$ characterising the diffusion in the chemical metric from the slope of the graphs in figure 4, from which we obtained $d_w^{l(2)} = 2.28 \pm 0.05$. From figures 3 and 4, we see that by increasing the number of steps N_1 of the RW that generates the cluster, the exponents $d_w^{(2)}$ and $d_w^{l(2)}$ decrease and converge to their asymptotic value. An alternative way to calculate the above exponents which yield similar results is to calculate the local derivative [21] of the functions $\langle R^2(N_2)\rangle$ and $\langle l(N_2)\rangle$.

The values for $d_w^{(2)}$ and $d_w^{l(2)}$ can be used to calculate the fracton dimensionality $d_s^{(2)}$ of the RW cluster generated in $d = 3$ using the relation [1, 6]

$$d_s^{(2)} = \frac{2d_f^{(2)}}{d_w^{(2)}} = \frac{2d_l^{(2)}}{d_w^{l(2)}}. \tag{7.1}$$

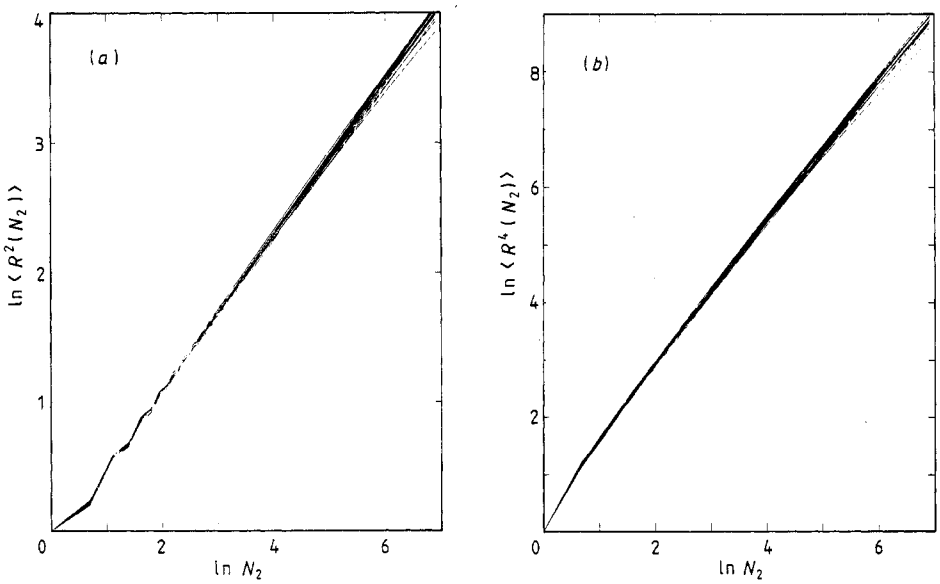

Figure 3. Plot of the mean square displacement (a) and the mean fourth moment of the displacement (b) of a RW of N_2 steps performed on a cluster generated by a RW of N_1 steps ($N_1 = 200, 400, 600, 800, 1000, 2000, 3000, 4000, 5000, 10\,000$ and $15\,000$) as a function of N_2 on a double logarithmic scale. From the slopes of these figures, we obtain $d_w^{(2)} = 3.45 \pm 0.10$.

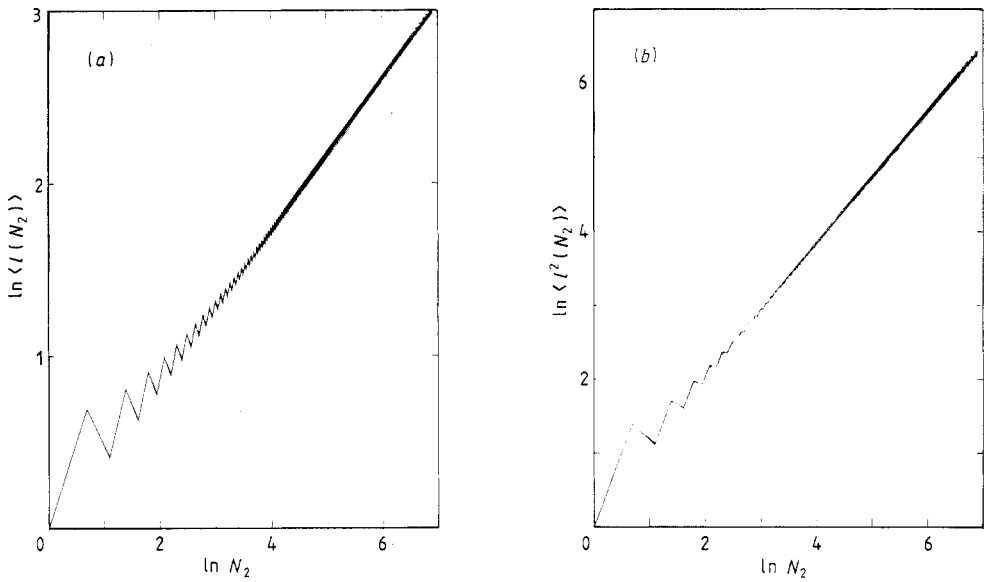

Figure 4. Plot of the mean chemical displacement (a) and the mean square chemical displacement (b) of a RW of N_2 steps performed on a cluster generated by a RW of N_1 steps as a function of N_2 on a double logarithmic scale. The values of N_1 and N_2 are given in the caption of figure 3. From the slopes of these figures, we obtain $d_w^{l(2)} = 2.28 \pm 0.05$.

From (7.1), we obtain $d_s^{(2)} = 2 \times 2/3.45 \simeq 2 \times 1.29/2.28 \simeq 1.15 \pm 0.02$. We also calculated $d_s^{(2)}$ independently using the fact that the probability of the RW on a cluster to return to the origin after N_2 steps scales as [1]

$$P_0(N_2) \sim N_2^{-d_s^{(2)}/2}. \tag{7.2}$$

Measuring the probability to return to the origin we obtained $d_s^{(2)} = 1.14 \pm 0.02$, which is in very good agreement with the above result. This result is inconsistent with the suggested extension [22, 23] of the Alexander-Orbach conjecture [1] that $d_s = \frac{4}{3}$ and the Aharony-Stauffer conjecture [24] that $d_s = 2d_f/(d_f+1) = \frac{4}{3}$ for random homogeneous fractals.

The values for $d_w^{(2)}$ and $d_w^{l(2)}$ can be used to calculate the exponent $\tilde{\nu}$ characterising the relation between the chemical and geometrical metrics in the cluster, using the relation

$$\tilde{\nu} = d_w^{l(2)}/d_w^{(2)}. \tag{7.3}$$

Substituting the above values for $d_w^{(2)}$ and $d_w^{l(2)}$ we obtain $\tilde{\nu} = 0.67 \pm 0.01$, which is in close agreement with the result obtained from static calculations of $\tilde{\nu}$ described in § 4. We find that this ratio depends only very weakly on N_1 and N_2, in contrast to the dependence of $\tilde{\nu}$ on N_1 in the static calculations. This may be explained by noting that the effects caused by increasing N_1 in the dynamical measurements are cancelled when one uses the ratio equation (7.3).

In § 6 a scaling function equation (6.6) was suggested to describe a RW of N_2 steps performed on a RW cluster of N_1 steps. One may test this scaling form numerically. We divide our ensembles into two groups. The first group contains the cluster ensembles

in which N_1 is smaller than N_2 ($N_2 = 1000$; $N_1 = 200$, 400, 600, 800, 1000) and the second group contains the cluster ensembles characterised by $N_1 \geqslant N_2$ ($N_2 = 1000$; $N_1 = 1000$, 2000, 3000, 4000, 5000, 10 000). This distinction is due to the fact that the diffusion exponents characterising clusters in the first group are greater than those characterising clusters in the second group. In figure 5(a) we present a plot of $\langle R_2^2(N_1, N_2)\rangle / N_1$ against $N_2^{d_w^{(2)}} / N_1$ for the first group using $d_w^{(2)} = 3.4$, and in figure 5(b) we plot the same function for the second group using $d_w^{(2)} = 3.35$. In both figures, we see that all numerical data can be fitted into the same curve when using the corresponding value of $d_w^{(2)}$. This is an independent method to determine $d_w^{(2)}$ since only for the correct $d_w^{(2)}$ does the numerical data collapse into one curve. We also checked the scaling function (6.8) suggested for the chemical metric. In this case, we divide the ensembles into the same two groups as before. In figure 6(a), we present a plot of the function $\langle l_2(N_1, N_2)\rangle / N_1$ against $N_2^{d_w^{l(2)}} / N_1$ for clusters of the first group using $d_w^{l(2)} = 2.33$, and in figure 6(b) we plot the same function for clusters which belong to the second group using $d_w^{l(2)} = 2.28$. In both figures, we see a good scaling fitting†.

We also tested the scaling function for the RW cluster generated in $d = 2$. In figures 7(a) and 7(b), we plot the function $\langle R_2^2(N_1, N_2)\rangle / N_1$ against $N_2^{d_w^{(2)}} / N_1$ for these clusters. As in $d = 3$, we divide the cluster ensembles into two groups. In figure 7(a) we plot the function for the clusters characterised by $N_1 \leqslant N_2$ using $d_w^{(2)} = 2.6$ and in figure 7(b) we plot the same function for clusters characterised by $N_1 \geqslant N_2$ using $d_w^{(2)} = 2.45$. The diffusion exponent $d_w^{(2)}$ is seen to decrease with increasing N_1. The values found

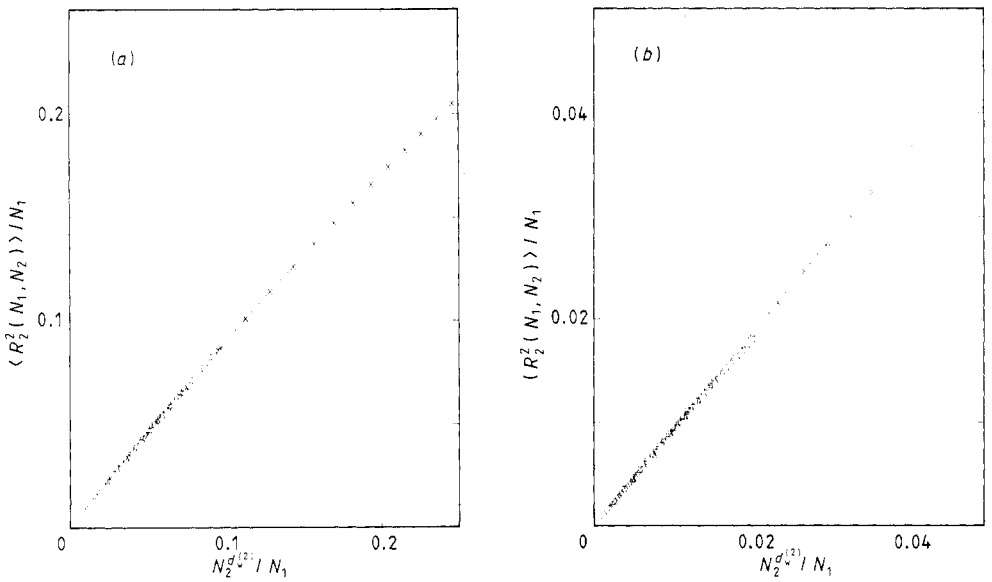

Figure 5. A plot of the scaling function $\langle R_2^2(N_1, N_2)\rangle / N_1$ as a function of $N_2^{2/d_w^{(2)}} / N_1$ for a RW of N_2 steps performed on clusters generated by a RW of N_1 steps. In a three-dimensional space (a) for $N_1 \leqslant N_2^{\max} = 1000$, $N_1 = 200$ (×), $N_1 = 400$ (△), $N_1 = 600$ (▽), $N_1 = 800$ (□), $N_1 = 1000$ (+), using $d_w = 3.4$ and in (b) for $N_1 \geqslant N_2^{\max}$ ($N_2^{\max} = 1000$, $N_1 = 1000$ (×), $N_1 = 2000$ (△), $N_1 = 3000$ (▽), $N_1 = 4000$ (□), $N_1 = 5000$ (+), $N_1 = 10\,000$ (◇)) using $d_w^{(2)} = 3.35$.

† Changing $d_w^{(2)}$ or $d_w^{l(2)}$ by ±0.05 does not lead to a collapse of the data into one curve.

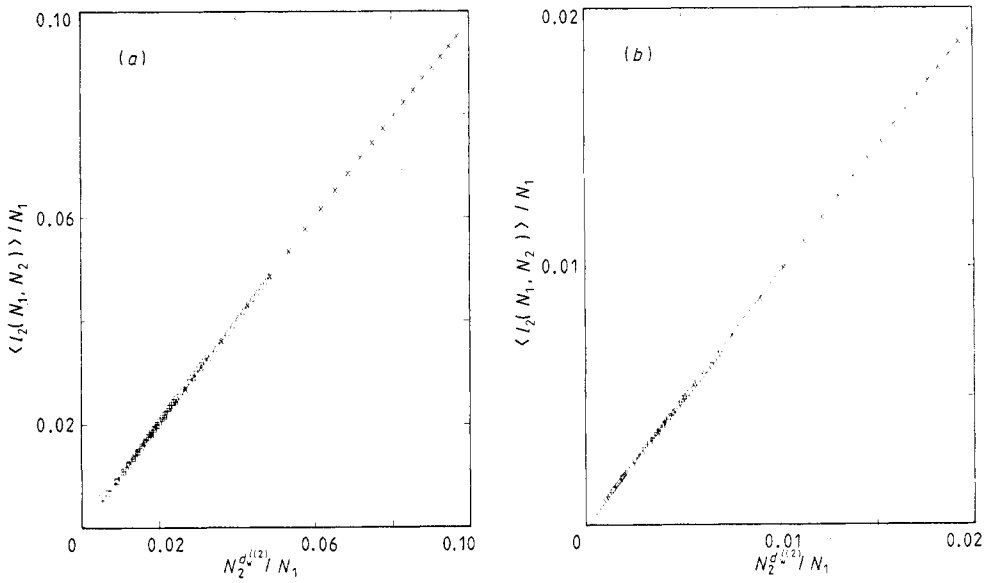

Figure 6. A plot of the scaling function $\langle l_2(N_1, N_2)\rangle / N_1$ as a function of $N_2^{2/d_w^l} / N_1$ for RW of N_2 steps performed on a RW of N_1 steps in a three-dimensional space. The curves in (*a*) are for $N_1 \leqslant N_2^{\max}$ (the values and symbols are as described for figure 5) using $d_w^{l(2)} = 2.33$ and in (*b*) for $N_1 \geqslant N_2^{\max}$ using $d_w^{l(2)} = 2.28$.

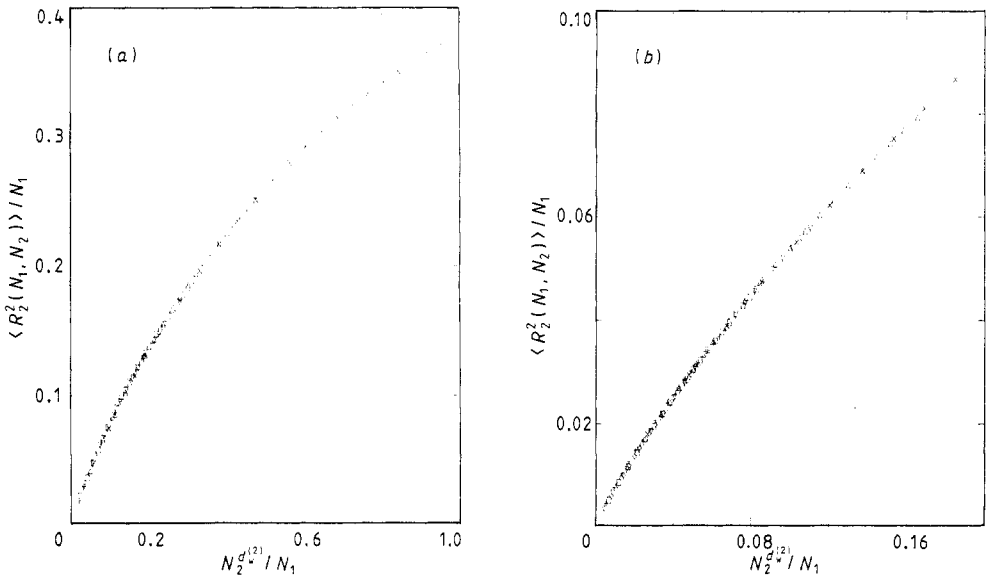

Figure 7. A plot of the scaling function $\langle R_2^2(N_1, N_2)\rangle / N_1$ against $N_2^{2/d_w^{(2)}} / N_1$ for RW performed on clusters generated by RW on a two-dimensional square lattice. We plot in (*a*) the curves for the clusters characterised by $N_1 \leqslant N_2^{\max}$ ($N_1^{\max} = 1000$, $N_1 = 200$ (×), $N_1 = 400$ (△), $N_1 = 600$ (▽), $N_1 = 800$ (□), $N_1 = 1000$ (+)) using $d_w^{(2)} = 2.6$. We plot in (*b*) the curves for the clusters characterised by $N_1 \geqslant N_2^{\max}$ ($N_2^{\max} = 1000$, $N_1 = 100$ (×), $N_1 = 2000$ (△), $N_1 = 3000$ (▽), $N_1 = 4000$ (□), $N_1 = 5000$ (+), $N_1 = 6000$ (◇)) using $d_w^{(2)} = 2.45$.

for $d_w^{(2)}$ are in close agreement with the result obtained by Banaver *et al* [10] from measuring the cluster resistivity. They found $\tilde{\zeta} = 0.5$ for the resistivity exponent of the RW cluster in $d = 2$. Using the Einstein relation (5.4), we obtained $d_w^{(2)} \simeq 2.5$. We believe that both results are due to finite-size effects, and in the limit $N_1 \to \infty$, the diffusion exponent will decrease to the value $d_w^{(2)} = 2$. A tendency in this direction can be seen from the fact that $d_w^{(2)}$ decrease monotonically as we increase N_1.

8. Discussion

In this paper we have studied the static and dynamical properties of clusters generated by a random walk. This random aggregate was defined in two ways, which were shown to coincide when the cluster is generated in space with $d \geqslant 2$. When the cluster generated in space with $d \leqslant 2$, it is space filling in the limit $N_1 \to \infty$, from which it follows that its properties are similar to the space properties. When the cluster is generated in $d \geqslant 4$, one can use the fact that the loops in the cluster are negligible upon scaling. Thus, the cluster has the same properties as the cluster generated by SAW.

The interesting case is the cluster generated in $d = 3$. On the one hand it contains loops in all length scales, and on the other hand the deadends are negligible. The importance of the loops is indicated by the fact that the cluster is characterised by $\tilde{\nu} \simeq \frac{2}{3}$ whereas for SAW clusters in $d = 3$, $\tilde{\nu} = \nu \simeq \frac{3}{5}$. This occurs because the loops create new paths in the cluster. Thus the path winding decreases and $\tilde{\nu}$ increases. This result for $\tilde{\nu}$ in $d = 3$ and the values of $\tilde{\nu}$ in $d = 2$ and $d = 4$ (see table 1) suggest the conjecture that $\tilde{\nu} = 2/d$ for RW clusters. We presented a Flory-type approximation to support this conjecture. However, this result has to be checked carefully in order to determine whether it is only a good approximation or an exact result.

Analysing our data for calculating the dynamical exponents we see that, by increasing the number of steps of the RW that generates the cluster, its fractal dimension and its chemical dimension increase, while the exponents characterising the dynamic properties of the cluster such as $d_w^{(2)}$ and $d_w^{l(2)}$ decrease. Although we have used very long RW to generate the clusters (up to 15 000 steps in $d = 3$), we still find small changes in the values of the exponents as N_1 increases. However, it seems that our estimations are very close to the asymptotic values.

For clusters without loops, it has been proven that $d_w^l = d_l + 1$, whereas in clusters which contain loops, $d_w^l \leqslant d_l + 1$. The difference $(d_l + 1) - d_w^l \equiv \Delta$ is a measure of the contribution of loops to the transport properties of the cluster. From the above discussion, it follows that by increasing N_1 the difference Δ increases, from which we conclude that the loops in the cluster become more dominant as we increase N_1. Our estimate for the asymptotic value is $\Delta \simeq 0.1$, which is close to the value $\Delta \simeq 0.12$ in percolation clusters in $d = 3$. However, it is interesting to compare this result with the value $\Delta \simeq 0.27$ characterising loops in the $d = 3$ Sierpinski gasket (SG). This indicates that the effect of loops in the Sierpinski gasket is significantly larger than in percolation and RW clusters.

Since the deadends in the RW cluster are negligible, it is interesting to compare the exponents characterising RW clusters with the exponents which characterise the backbone of the percolation cluster in $d = 3$. These two fractals have the same fractal dimension ($d_f = 2$) and also the diffusion exponent characterising the RW cluster, $d_w^2 \simeq 3.45 \pm 0.05$, is close to the diffusion exponent characterising the percolation backbone, $d_w \simeq 3.2$. However, the exponent $\tilde{\nu}$ is different. In RW clusters $\tilde{\nu} \simeq 0.66$, whereas

in the percolation backbone $\tilde{\nu} = 0.75$. Thus the properties of the cluster in the chemical metric are different.

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