Multifractal behavior of linear polymers in disordered media

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The scaling behavior of linear polymers in disordered media modeled by self-avoiding random walks (SAWs) on the backbone of two- and three-dimensional percolation clusters at their critical concentrations pc is studied. All possible SAW configurations of N steps on a single backbone configuration are enumerated exactly. We find that the moments of order q of the total number of SAWs obtained by averaging over many backbone configurations display multifractal behavior; i.e., different moments are dominated by different subsets of the backbone. This leads to generalized coordination numbers μq and enhancement exponents γq, which depend on q. Our numerical results suggest that the relation μ1 = pμ between the first moment μ1 and its regular lattice counterpart μ is valid.

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I. INTRODUCTION

The question of how linear polymers behave in a disordered medium has attracted much attention in recent years. The problem is not only interesting from a theoretical point of view, but may also be relevant for understanding transport properties of polymeric chains in porous media, such as enhanced oil recovery, gel electrophoresis, gel permeation chromatography, etc. [1–4]. In this context, it is useful to learn about the static or conformational properties of linear chains, modeled by self-avoiding walks (SAWs), in the presence of quenched disorder, e.g., how the surrounding structural disorder influences their spatial configuration. As a quite general model of a random medium, percolation [5–8] may be considered the paradigm for a broad class of disordered systems and has therefore been mostly used so far.

We are interested in how the statistical behavior of SAWs on percolation clusters at criticality (p = pc) differs from their behavior on regular lattices. While the values of the exponents for SAWs on regular lattices are well established [1,9–12], there is no complete agreement about their values on percolation clusters at pc [13,14]. Here we study (i) the so-called effective coordination number of the cluster, where contradicting results have been reported using different numerical techniques. Next we consider (ii) the enhancement exponent γ and (iii) the exponents νr and νl, characterizing the end-to-end distance of SAWs in the r- and l-space metrics. Finally, we determine (iv) the values of the critical exponents describing the corresponding structural distribution functions.

We concentrate on SAWs on percolation clusters at pc in two and three dimensions. In the literature, two distinct methods have been used for evaluating SAWs: Exact enumeration (EE) and Monte Carlo (MC) simulation. In the EE technique, all SAW configurations on a given cluster are taken into account, but only relatively short chains can be evaluated. In a MC simulation, longer chains can be studied, but inherently the ensemble of configurations remains incomplete. Here we use the EE technique in combination with an appropriate finite-size scaling procedure to determine the relevant exponents. Since ‘‘infinitely’’ long chains can only exist on the backbone of the cluster, where dangling ends are absent on all length scales, we study the SAWs directly on the backbone. This enables us to generate longer chains on a given cluster and to average over a larger set of different cluster configurations.

Specifically, we enumerate all possible SAW configurations of N steps for a single backbone and study different moments of the total number of SAWs and their end-to-end distance by averaging over many different backbone configurations. Our analysis shows that the critical exponents νr and νl do not depend on the order q of the moments, while the enhancement exponents and the effective coordination numbers do depend on q, leading to multifractal behavior. In particular, we find that the first moment of the effective coordination number satisfies μ1 = pμ, where μ is the effective coordination number of the underlying regular lattice, resolving previous controversies. The mean structural distribution functions for the end-to-end distance after N steps, both in Euclidean and topological space, are obtained numerically, supporting the expected scaling forms [15,16].

The paper is organized as follows: In Sec. II, we briefly review the main relevant properties of SAWs on regular lattices to illustrate the different numerical procedures employed in this work. In Sec. III, we present results for the total number and the mean end-to-end distance of SAWs on the backbone of the incipient percolation cluster. The corresponding distribution functions of the end-to-end distance and their scaling behavior, in Euclidean and topological space, are also discussed. Finally, in Sec. IV we summarize our main results.

II. SAWS ON REGULAR LATTICES REVIEWED

In this section, we illustrate the different numerical techniques we use in the following sections by briefly reviewing the main results for SAWs on regular lattices. The main idea is to show that our finite-size scaling, employed in the later
sections, enables us to obtain quite accurate estimates for the critical exponents based on EE results for relatively short chains. Here we consider the case $d=2$, which is particularly suitable since many results are known exactly.

### A. Total number of SAW configurations $C_N$

The total number $C_N$ of SAW configurations of $N$ steps behaves as [11]

$$C_N = A \mu^N N^{\gamma-1},$$  

(1)

where $\mu$ is the effective coordination number of the lattice, $\gamma$ is the universal enhancement exponent, and $A$ is a constant. To determine $\mu$, $\gamma$, and $A$, we choose to study the behavior of the quantity

$$\ln \frac{C_N}{N} = \frac{\ln A}{N} + \ln \mu + (\gamma - 1) \ln N$$  

(2)

as a function of $N$. Figure 1 shows that for the square lattice, the values for $\mu$ and $\gamma$ obtained by fitting the EE data using Eq. (2) agree well with the accepted values reported in the literature (see Table I).

### B. Mean end-to-end distance and structural distribution function

The root-mean-square end-to-end distance of SAWs of $N$ steps, $r(N)=\langle r^2(N) \rangle^{1/2}$, averaged over all possible SAW configurations behaves as

$$f(x) \propto \begin{cases} x^{g_1+d}, & x \ll 1 \\ x^{g_2+d} \exp(-c x^\delta), & x \gg 1, \end{cases}$$  

(5)

where $g_1=1/(2 \nu_f)$ [21], $g_2=\delta(d(1/2)-1)$ [22], and $\delta=1/(1-\nu_f)$ [10]. Values for these exponents are summarized in Table I. We have verified these predictions by enumerating all SAW configurations for $N=23$ and 24 and calculating the corresponding distributions $P(r,N)$, from which we have extracted the different exponents (see Fig. 3). We show that a more accurate determination of the exponent $g_2$ compared to a simple fit using Eq. (5) can be obtained by employing a specific numerical procedure described in Appendix A (see inset of Fig. 3). The obtained values are in agreement with the theoretical predictions (see Table I).

### III. SAWS ON THE BACKBONE OF THE INCipient PERCOLATION CLUSTER

Next, we consider SAWs on the incipient percolation cluster by generating all SAW configurations directly on the backbone of the cluster. We obtain the backbone of a given cluster grown by the Leath algorithm [23,24] by randomly
the chemical distance $l$ between two backbone sites separated by the Euclidean distance $r$ increases with $r$ as [27,28]

$$l \propto r^{d_{\min}},$$

where $d_{\min} = 1.1306 \pm 0.0003$ in $d = 2$ [29] and $d_{\min} = 1.374 \pm 0.004$ in $d = 3$ [30]. Thus Eq. (6) yields the scaling relation between the two metrics, which will be used in what follows. Numerically it is found that data obtained in $l$ space show less fluctuations (cf., e.g., [15]). Therefore more accurate estimates for many characteristic quantities (such as critical exponents) in $r$ space can be determined by studying the corresponding quantity in $l$ space and transforming it to $r$ space. For example, the fractal dimension of the backbone in $l$ space is $d_{\text{perc}}^B = 1.45 \pm 0.01$ in $d = 2$ and $d_{\text{perc}}^B = 1.36 \pm 0.02$ in $d = 3$. Using Eq. (6), this leads to the values $d_{f}^B = d_{\text{perc}}^B d_{\min} = 1.64 \pm 0.02$ and $d_{f}^B = 1.87 \pm 0.03$ in $r$ space, respectively [26].

A. Total number of SAW configurations: Multifractality

Due to the disordered structure of the clusters, the total number $C_{N,B}$ of SAW configurations that are generated on a single backbone, with the seed $S$ of the cluster as the starting point, fluctuates strongly among different backbone configurations. To characterize these fluctuations, we study the moments $\langle C_{N,B}^q \rangle$. A similar study on percolation clusters at criticality has been performed for “ideal” chains; i.e., chains that can intersect themselves. This model leads to a non-trivial dependence on $q$ [31].

In generalizing Eq. (1), we make the ansatz

$$\langle C_{N,B}^q \rangle^{1/q} = A_q \mu_q N^\gamma_q^{-1},$$

where $\mu_q$ are the generalized effective coordination numbers of the backbone and $\gamma_q$ the generalized enhancement exponents. Results for different values of $q$ are shown in Figs. 5(a) and 5(b) for the square and simple cubic lattice, respectively, employing the numerical procedure described in Sec. II A. The values for $\mu_q$ and $\gamma_q$ are displayed in Fig. 6 for $d = 2$, clearly revealing a dependence on $q$, reminiscent of a multifractal behavior. For large negative values of $q$, backbone configurations with a small number of SAW configurations $C_{N,B}$ are singled out in the averaging procedure. We find that $\mu_q \to 1$ and $\gamma_q \to 1$ for $q \to -\infty$, pointing to rare configurations of backbones with an almost linear shape. On the contrary, for large values of $q$ the averaging procedure emphasizes backbone configurations with a large number of SAW configurations $C_{N,B}$. Since these backbones are the most compact ones, $\mu_q$ and $\gamma_q$ are strongly enlarged. Figure 6 seems to suggest that the structure of the most compact backbone differs distinctively from the structure of a regular square lattice, as $\lim_{q \to -\infty} \mu_q \approx 1.9$, which is well below the value for $\mu$ on the regular square lattice, and $\lim_{q \to -\infty} \gamma_q \approx 1.7$ is well above the value for $\gamma$ on the regular square lattice.

These results resolve earlier controversies regarding the values for both $\mu$ and $\gamma$ for percolation obtained from MC simulations and by EE techniques. For the square lattice, for example, the values $\mu_{\text{perc}}(\text{EE}) = 1.53 \pm 0.05$ [32] and $\gamma_{\text{perc}}(\text{EE}) = 1.33 \pm 0.02$ [33] have been obtained from exact enumeration calculations, while from MC simulations the
values \( \mu_{\text{perc}}(\text{MC}) = 1.459 \pm 0.003 \) and \( \gamma_{\text{perc}}(\text{MC}) = 1.31 \pm 0.03 \) [34] were determined. We find \( \mu_1 = 1.565 \pm 0.005 \), \( \gamma_1 = 1.34 \pm 0.05 \), and \( \mu_2 = 1.546 \pm 0.005 \), \( \gamma_2 = 1.26 \pm 0.05 \), corresponding to the EE and MC results, respectively. This can be understood by noting that EE calculations yield by definition the whole ensemble (the so-called ‘‘annealed’’ average), corresponding to the case \( q = 1 \); i.e., the normal arithmetic average. In contrast, MC simulations intrinsically sample only a small subset of all possible configurations, omitting rare configurations, yielding ‘‘typical’’ subsets of the ensemble (the so-called ‘‘quenched’’ average). This quenched average is usually described by a logarithmic average, i.e., \( \langle C_{N,B}\rangle_{\text{perc}} = \exp \langle \ln C_{N,B}\rangle \), and is equivalent to the limit \( q \to 0 \) of Eq. (7); i.e., \( \lim_{q \to 0} \langle C_{N,B}\rangle_{\text{perc}}^{1/q} = \exp \langle \ln C_{N,B}\rangle \). Indeed, our results are in excellent agreement, in both \( d = 2 \) and \( d = 3 \), with the relation

\[
\mu_1 = p_c \mu,
\]

where \( \mu \) is the effective coordination number of the underlying regular lattice, \( p_c = 0.5927460 \) for the square lattice [35] and \( p_c = 0.311605 \) for the simple cubic lattice [36]. This relation, which was originally suggested in the form \( \mu_{\text{perc}} = p_c \mu \) [34], could not be confirmed earlier on because of the different values obtained for \( \mu_{\text{perc}} \). Because of the possible existence of rare events playing a dominant role in the average procedure, we have performed a detailed analysis of our numerical data to confirm that we have considered a sufficiently large set of cluster configurations (cf. Appendix B).

**B. Mean end-to-end distances and structural distribution functions**

Next we study the scaling behavior of the distribution functions for the end-to-end distance, \( \langle P_B(l,N) \rangle \) and \( \langle P_B(r,N) \rangle \), averaged over many backbone configurations, where \( P_B(l,N)dl \) is the probability that after \( N \) steps, the chemical end-to-end distance of a chain on a single backbone is between \( l \) and \( l + dl \), and \( P_B(r,N)dr \) is the analogous quantity in \( r \) space. These distribution functions are expected to obey scaling forms similar to the one valid on regular lattices, Eq. (4), with the corresponding scaling exponents [15]. The mean chemical end-to-end distance \( \langle \bar{l}(N) \rangle \) and the root mean-square Euclidean end-to-end distance \( \langle \bar{r}(N) \rangle \) scale with \( N \) as

\[
\langle \bar{l}(N) \rangle \propto N^{\nu_l},
\]

\[
\langle \bar{r}(N) \rangle \propto N^{\nu_r},
\]

respectively. The first average is performed over all SAW configurations on a single backbone; the second average is carried out over many backbone configurations. Following Eq. (6), the exponents \( \nu_l \) and \( \nu_r \) are related to each other by \( \nu_r = \nu_l / d_{\text{min}} \). The numerical results for \( \nu_l \) and \( \nu_r \) obtained by the successive slopes technique discussed in Sec. II B for regular lattices are reported in Table II. As an example, Fig. 7 shows the determination of \( \nu_l = d = 3 \).

Accordingly, the scaling variable in chemical space is \( l/N^{\nu_l} \), and the mean structural distribution function, averaged over many backbone configurations, has the form

\[
\langle P_B(l,N) \rangle \propto f_1(l/N^{\nu_l})
\]

with the scaling function

\[
f_1(x) \propto \begin{cases} \frac{1}{x} \exp(-c_d x^{\theta_1}), & x < 1 \\ x^{\theta_2} \exp(-c_d x^{\theta_1}), & x \geqslant 1 \end{cases}
\]
TABLE II. Structural parameters for SAWs on the backbone of percolation clusters at criticality in \( d = 2 \) and \( d = 3 \), on the square and simple cubic lattice, respectively. The values for \( \nu_r \), obtained directly from the numerical data, are in agreement with the more precise values obtained from the relation \( \nu_r = \nu_l/d_{\text{min}} \). The values for \( g_1' = g_1'd_{\text{min}} \) are also in good agreement with the corresponding values obtained directly from the data. The numerical values for the exponents \( g_Z^1 \) and \( g_Z^2 \) were determined using the procedure described in Appendix A. Note that there is no simple relation between \( g_Z^1 \) and \( g_Z^2 \); i.e., \( g_Z^1 \neq g_Z^2 d_{\text{min}} \). The values of \( \delta_l \) and \( \delta_r \) are consistent, within the present accuracy, with the expressions \( \delta_l = 1/(1 - \nu_l) \) and \( \delta_r = 1/(1 - \nu_r) \).

<table>
<thead>
<tr>
<th>( d )</th>
<th>( \gamma_1 )</th>
<th>( \gamma_0 )</th>
<th>( \mu_1 )</th>
<th>( \mu_0 )</th>
<th>( \nu_l )</th>
<th>( \nu_r ) (directly from data)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.34±0.05</td>
<td>1.26±0.05</td>
<td>1.56±0.005</td>
<td>1.456±0.005</td>
<td>0.89±0.01</td>
<td>0.778±0.006</td>
</tr>
<tr>
<td>3</td>
<td>1.29±0.05</td>
<td>1.19±0.05</td>
<td>1.462±0.005</td>
<td>1.317±0.005</td>
<td>0.910±0.005</td>
<td>0.662±0.006</td>
</tr>
</tbody>
</table>

\( \nu_l = \nu_l/d_{\text{min}} \)

\( g_1' = g_1'd_{\text{min}} \)

\( g_2' = 1.6\pm0.16 \)

\( g_2' = 1.26\pm0.18 \)

\( \delta_l = 9.5\pm0.5 \)

\( \delta_r = 4.85\pm0.20 \)

Equivalently, in \( r \) space, the scaling variable is \( r/N^{\nu_r} \), and one has

\[
\langle P_B(r,N) \rangle \sim \frac{1}{r} f(r/N^{\nu_r})
\]

with

\[
f_r(x) = \begin{cases} x^{g_1'+d_1'}, & x \ll 1 \\ x^{g_2'+d_1'} \exp(-c_{d,r}x^{\delta_r}), & x \gg 1. \end{cases}
\]

Both distribution functions are normalized according to \( \int_0^\infty d\langle P_B(1,N) \rangle = 1 \) and \( \int_0^\infty d\langle P_B(r,N) \rangle = 1 \).

FIG. 7. The mean topological end-to-end distance \( \langle \bar{l}(N) \rangle \) versus \( N \) for SAWs on the backbone of critical percolation clusters in \( d = 3 \) averaged over \( 5 \times 10^4 \) backbone configurations. In the inset, the successive slopes \( \nu_r = \nu_l/d_{\text{min}} \) reported in Table II are used in the scaling variables. For the determination of the exponents \( g_1', g_2', g_1 \), and \( g_2 \) according to Eqs. (12) and (14), we use the previously reported values of the fractal dimensions \( d_1^F \) and \( d_2^F \) [26]. The exponents \( g_1' \) and \( g_1 \) can be estimated directly from the slope of \( f_j \) and \( f_r \) in the double logarithmic plots. Since \( g_1' \) and \( g_1 \) are related by \( g_1' = g_1'd_{\text{min}}^{-1} [15] \), a more precise estimate for \( g_1' \) can be derived from the estimate for \( g_1 \).

The determination of \( g_2' \) and \( g_2 \) is more difficult, since both exponents occur in the nondominant part and are masked by the exponential. Therefore it requires the use of the slightly more involved numerical procedure discussed in Appendix A (see the insets of Figs. 8 and 9 for \( d = 2 \) and \( d = 3 \), respectively). The numerical results we obtain for \( g_1', g_2', g_1 \), and \( g_2 \) are reported in Table II. Regarding the exponential factors, our results for the exponents \( \delta_l \) and \( \delta_r \) are consistent,

FIG. 8. Scaling plots of the distribution functions on the backbone in \( d = 2 \), for \( N = 39 \) and \( 40 \), averaged over \( 5 \times 10^3 \) configurations. (a) \( \langle P_B(1,N) \rangle \) versus \( l/N^{\nu_r} \): The dashed line has the slope 1.90 and corresponds to the ansatz Eq. (12) for \( x \ll 1 \); the continuous line is a fit with the ansatz Eq. (12) for \( x \gg 1 \), yielding \( g_2' = 1.4\pm0.4 \), \( \delta_l = 9.5\pm0.5 \), and \( \delta_r = 0.09\pm0.01 \). The inset shows \( l(\bar{P}_B(1,N)) = h_1^{g_1'+d_1'}(\Omega B)^{-1}\langle P_B(1,N) \rangle \exp(b_1 l/N^{\nu_r}) \) versus \( l/N^{\nu_r} \), with our estimate of the crossover value \( z_r = 0.21 \), according to the procedure described in Appendix A, yielding the more precise estimate \( g_2' = 1.46\pm0.4 \), \( \delta_l = 4.9\pm0.3 \), and \( \delta_r = 0.79\pm0.10 \). The inset shows \( r(\bar{P}_B(r,N)) = h_1^{g_1'+d_1'}(\Omega B)^{-1}r(\bar{P}_B(r,N)) \exp(b_1 l/N^{\nu_r}) \) versus \( l/N^{\nu_r} \), with our estimate of the crossover value \( z_r = 0.25 \), according to the procedure described in Appendix A, yielding the more precise estimate \( g_2' = 2.89\pm0.15 \). (b) \( r(\bar{P}_B(r,N)) \) versus \( l/N^{\nu_r} \): The dashed line has the slope 1.9 and corresponds to the ansatz Eq. (12) for \( x \ll 1 \); the continuous line is a fit with the ansatz Eq. (12) for \( x \gg 1 \), yielding \( g_2' = 1.5\pm0.2 \), \( \delta_l = 9.8\pm0.5 \), and \( \delta_r = 0.09\pm0.01 \). The inset shows \( r(\bar{P}_B(r,N)) = h_1^{g_1'+d_1'}(\Omega B)^{-1}r(\bar{P}_B(r,N)) \exp(b_1 l/N^{\nu_r}) \) versus \( l/N^{\nu_r} \), with our estimate of the crossover value \( z_r = 0.25 \), according to the procedure described in Appendix A, yielding the more precise estimate \( g_2' = 2.9\pm0.15 \).
FIG. 9. Scaling plots of the distribution functions on the backbone in $d = 3$, for $N = 39$ and $40$, averaged over $5 \times 10^3$ configurations. (a) $l(P_b(l,N))$ versus $l/N^{\nu_1}$. The dashed line has the slope 2.02 and corresponds to the ansatz Eq. (12) for $x \ll 1$; the continuous line is a fit with the ansatz Eq. (12) for $x \gg 1$, yielding $g_2^l = 1.3 \pm 0.6$, $\delta_b = 12.0 \pm 0.5$, and $c_{3j} = 0.06 \pm 0.01$. The inset shows $l(P_b(l,N)) = b_1 l(N)^{\delta_b} \exp(b_2 l(N)^{\delta_b})/l/N^{\nu_1}$ versus $b_1 l(N)^{\delta_b} l/N^{\nu_1}$, with our estimate of the crossover value $z_c = 4.0$, according to the procedure described in Appendix A, yielding the more precise estimate $g_2^l + d_b^l = 3.31 \pm 0.15$ (continuous line). (b) $r(P_b(r,N))$ versus $r/N^{\nu_r}$; the dashed line has the slope 2.78 and corresponds to the ansatz Eq. (14) for $x \ll 1$; the continuous line is a fit with the ansatz Eq. (14) for $x \gg 1$, yielding $g_2^r = 2.3 \pm 0.6$, $\delta_r = 3.5 \pm 0.5$, and $c_{3j} = 0.08 \pm 0.10$. The inset shows $r(P_b(r,N)) = b_1' r(N)^{\delta_r} \exp(b_2' r(N)^{\delta_r})/r/N^{\nu_r}$ versus $b_1' r(N)^{\delta_r} r/N^{\nu_r}$, with our estimate of the crossover value $z_c = 0.5$, yielding the more precise estimate $g_2^r + d_r^b = 4.83 \pm 0.15$ (continuous line).
and from the second moment
\[
\int_0^\infty r^2 P(r,N)dr=r^2(N)\equiv N^2 r^2. \tag{A4}
\]

Upon integration of Eqs. (A3) and (A4), one gets the exact relations
\[
B=\frac{1}{\Omega} \left[ \frac{1}{\delta b^{(g_2+d)/\delta}} \Gamma \left( \frac{g_2+d}{\delta}, b z^\delta \right) + \frac{g_1+d}{g_1+d} \right]^{-1}, \tag{A5}
\]
where \( \Gamma(u,z) \) is the incomplete gamma function, and
\[
\Omega B \left[ \left( \frac{g_2+d+2}{\delta}, b z^\delta \right) + \frac{g_1+d+2}{g_1+d+2} \right] = 1. \tag{A6}
\]

Thus by plotting the distribution function in the case \( x>z \) as \( y=b^{(g_2+d)/\delta} \Omega B^{-1} r P(r,N) \exp[(b^{(1/d)}r/N^q)^\delta] \) versus \( b^{1/d} r/N^q \) in a double logarithmic plot, the exponent \( g_2 \) can be read off from the relation \( y=x^{g_2+d} \) and adjusted until the above relations Eqs. (A5) and (A6) are satisfied. This method yields much more accurate results than by directly fitting the distribution function itself. The accuracy of the result can be assessed by plotting \( y= -\ln[b^{(g_2+d)/\delta} \Omega B^{-1} r P(r,N)(b^{(1/d)}r/N^q)^{-g_2+d}] = b(r/N^q)^\delta \) versus \( b^{1/d} r/N^q \) in a double logarithmic plot, from which the exponent \( \delta \) can be determined and compared with the expected value \( \delta=1/(1-\nu_p) \). The procedure can be extended straightforwardly to study the distribution functions \( P_B(l,N) \) and \( P_B(q,r,N) \) of SAWs on the backbone of critical percolation clusters.

APPENDIX B: GENERALIZED AVERAGING PROCEDURE

To obtain an estimate of whether the ensemble \( B \) of backbone configurations considered is sufficiently large to get convergent results, we analyze the data by a generalized averaging procedure as follows: The total ensemble \( B \) containing \( n_{\text{tot}} \) backbone configurations is divided into subsets \( B_i \) containing \( n_{\text{eff}} \) configurations each. The generalized average is then defined as
\[
\langle C_{N,B}(N) \rangle_{n_{\text{eff}}} = \left( \frac{1}{n_{\text{eff}}} \sum_{i=1}^{n_{\text{eff}}} \langle C_{N,B}(N) \rangle_i \right)^{1/q}. \tag{B1}
\]

The obtained results \( \langle C_{N,B}(N) \rangle_{n_{\text{eff}}} \) depend sensitively on the diffusion length
\[
\langle \mu_{n_{\text{eff}}} \rangle_{n_{\text{eff}}} = \left( \frac{1}{n_{\text{eff}}} \sum_{i=1}^{n_{\text{eff}}} \langle \mu_{n_{\text{eff}}} \rangle_i \right)^{1/q}, \tag{B2}
\]

FIG. 10. The effective coordination numbers \( \mu_{q,n_{\text{eff}}} \) (circles) and the enhancement exponents \( \gamma_{q,n_{\text{eff}}} \) (squares) of SAWs on the backbone in \( d=2 \) for (a) \( q=1 \) and (b) \( q=2 \) versus the effective ensemble size \( n_{\text{eff}} \).

In Eq. (B2), the limiting case \( n_{\text{eff}}=1 \) corresponds to the limit \( q \rightarrow 0 \), while the usual average [cf. Eq. (7)] is recovered when \( n_{\text{eff}}=n_{\text{tot}} \). The results for the coordination numbers \( \mu_{q,n_{\text{eff}}} \) and enhancement exponents \( \gamma_{q,n_{\text{eff}}} \) are shown in Figs. 10(a) (for \( q=1 \)) and 10(b) (for \( q=2 \)). A dependence of these two values on \( n_{\text{eff}} \) indicates that the given ensemble is too small to obtain the asymptotic values. If, on the contrary, the ensemble of backbone configurations is sufficiently large, then \( \mu_{q,n_{\text{eff}}} \) and \( \gamma_{q,n_{\text{eff}}} \) no longer depend on \( n_{\text{eff}} \). For \( q=1 \), this seems to be the case when \( n_{\text{eff}}>10^3 \), and for \( q=2 \) when \( n_{\text{eff}} \gtrsim 10^4 \).


