Dependence of conductance on percolation backbone mass

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(Received 12 October 1999)

We study $\langle\sigma(M_B,r)\rangle$, the average conductance of the backbone, defined by two points separated by Euclidean distance r, of mass M_B on two-dimensional percolation clusters at the percolation threshold. We find that with increasing M_B and for fixed r, $\langle\sigma(M_B,r)\rangle$ asymptotically *decreases* to a constant, in contrast with the behavior of homogeneous systems and nonrandom fractals (such as the Sierpinski gasket) in which conductance increases with increasing M_B. We explain this behavior by studying the distribution of shortest paths between the two points on clusters with a given M_B. We also study the dependence of conductance on M_B above the percolation threshold and find that (i) slightly above p_c, the conductance first decreases and then increases with increasing M_B and (ii) further above p_c, the conductance increases monotonically for all values of M_B, as is the case for homogeneous systems.

PACS number(s): 64.60.Ak, 64.60.Fr, 05.45.Df

I. INTRODUCTION

There has been considerable study of the bond percolation cluster considered as a random-resistor network, with each occupied bond having unit resistance and nonoccupied bonds having infinite resistance [1–3]. In two dimensions, the configuration studied is typically an $L\times L$ lattice and the conductance is measured between two opposite sides which are assumed to have infinite conductance [4–16]. The backbone of the cluster is then defined as the set of bonds that are connected to the two sides having infinite conductance through paths that have no common bond.

At the percolation threshold, the backbone mass scales as $\langle M_B\rangle\sim L^{d_B}$ with $d_B=1.6432\pm0.0008$ [17] and in this ''bus bar'' geometry is strongly correlated with L. The average conductance of the backbone as a function of L has been studied extensively and has been found to scale as $\langle\sigma\rangle\sim L^{-\tilde{\mu}}$ with $\tilde{\mu}=0.9826\pm0.0008$ [17].

Recently, the distribution of masses of backbones defined by two *points*, i.e., backbones defined as the set of those bonds that are connected by paths having no common bonds to two points separated by distance r within an $L\times L$ lattice, has been studied [18]. This geometry has particular relevance to the oil industry where the oil field is represented by the percolation cluster and the two points represent the location of injection and production wells. One finds that when $r \ll L$, there is a very broad distribution of backbone masses for a given r. Figure 1 illustrates some typical percolation clusters and their backbones defined in this configuration. Because of the broad distribution of backbone masses we have the opportunity to study the conductance between these two points separated by a fixed distance r as a function of the mass of the backbone defined by these points.

One might expect that, for fixed r, the average conductance would *increase* with increasing backbone mass because there could be more paths through which current can flow. In fact, we find that the average conductance *decreases* monotonically with increasing backbone size, in contrast with the behavior of homogeneous systems and nonrandom fractals in which conductance increases. We explain our finding by first noting that the conductance is strongly correlated with the shortest path between the two points, and then studying the distribution of shortest paths along the backbone between the two points for a given M_B. This analysis extends recent studies of the distribution of shortest paths where no restriction on M_B is placed [19–22].

II. SIMULATIONS

Our system is a two-dimensional square lattice of side $L=1000$ with points A and B defined as $A=(L-r/2,\ 500)$, $B=(L+r/2,\ 500)$. For each realization of bond percolation on this lattice, if there is a path of connected bonds between A and B, we calculate (i) the length of the shortest path between A and B, (ii) the size of the backbone defined by A and B, and (iii) the total conductance between A and B. We perform 100 000 realizations at the percolation threshold, $p_c=0.5$, for each of 8 values of r (1, 2, 4, 8, 16, 32, 64, and 128). We bin these results based on the value of the backbone mass, M_B, by combining results for all realizations with $2^n<M_B<2^{n+1}$ and choosing the center of each bin as the value of M_B.

In Fig. 2(a), we plot the simulation results for the average conductance $\langle\sigma(M_B,r)\rangle$ and find that the conductance, in fact, *decreases* with increasing M_B. The decrease is seen more clearly in Fig. 2(b), in which we plot scaled values as discussed below.

III. SIERPINSKI GASKET

In nonfractal systems, the conductance increases as the mass of the conductor increases. We next consider the aver-

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age conductance on the Sierpinski gasket, a nonrandom fractal, the first three generations of which are illustrated in Figs. 3(a)–3(c). Because the Sierpinksi gasket is not translationally invariant, the analog of the average conductivity between two points in the percolation cluster is the conductivity averaged over all pairs of points separated by distance r. At each successive generation, there are two types of pairs: (i) pairs which correspond to pairs in the previous generation (e.g., A and B) and (ii) pairs which do not correspond to pairs in the previous generation (e.g., D and E). It is obvious that as we move from one generation to the next, the conductance between pairs of type (i) increases because there are more paths between the points than in the previous generation. On the other hand, the conductance between the pairs of type (ii) are lower on average than between the pairs present in the previous generation because on average the shortest path between the two points is longer than between the pairs in the previous generation. However, for any given r, the shortest path between any two points has a fixed upper bound independent of the generation. Due to this bound on the shortest path, the average conductivity increases with succeeding generations. This is shown in Fig. 3(d) which shows the average conductivity calculated exactly for generations 1 to 6 for $r=1$, 2 and 4.

IV. SHORTEST PATH DISTRIBUTION

In order to understand why the average conductance of the percolation backbone decreases with increasing M_B, we must (i) recognize that the conductance is strongly correlated with the shortest path [23] between the two points and (ii) study $P(\ell|M_B,r)$, the distribution of shortest paths between the two points for a given backbone mass. Hence we next create the $P(\ell|M_B,r)$ probability distribution, binning our results logarithmically by forming the average over samples centered at $\log_2\ell$.

Figure 4(a) shows the simulation results for $P(\ell|M_B,r)$ for $r=1$ for various backbone masses. The plots collapse, the only difference in the plots being the values of the upper cutoffs due to the finite backbone size. Figure 1 illustrates how the size of the backbone constrains the possible values of the shortest path. For all values of M_B, a section of each plot in Fig. 4(a) exhibits power law behavior. In Fig. 4(b), we show the distributions $P(\ell|M_B,r)$ for different r and a given M_B. In Fig. 4(c) we see that when scaled with $r^{d_{\min}}$ the plots collapse, so we can write $P(\ell|M_B,r)$ in the scaling form

$$P(\ell|M_B,r)\sim\frac{1}{r^{d_{\min}}}\left(\frac{\ell}{r^{d_{\min}}}\right)^{-\psi}. \quad (1)$$

An expression for ψ can be found by recognizing that we can write the well-studied distribution $P(\ell|r)$, the probability that the shortest path between two points separated by Euclidean distance r is ℓ, independent of M_B, as

$$P(\ell|r)=\int_{c_\ell}^{\infty}P(\ell|M_B,r)P(M_B|r)dM_B, \quad (2)$$

where (i) $P(M_B|r)$ is the distribution of backbone masses given distance r between the points which determine the backbone and (ii) c_ℓ is the lower cutoff on M_B given ℓ. $P(M_B|r)$ has the form [18]

$$P(M_B|r)\sim\frac{1}{r^{d_B}}\left(\frac{M_B}{r^{d_B}}\right)^{-\tau_B}, \quad [r\ll L], \quad (3)$$

where d_B is the backbone fractal dimension and