

The Directed Polymer — Directed Percolation Transition

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ABSTRACT

We study the relation between the directed polymer and the directed percolation models, for the case of a disordered energy landscape where the energies are taken from bimodal distribution. We find that at the critical concentration of the directed percolation, the directed polymer undergoes a transition from the directed polymer universality class to the directed percolation universality class. We also find that directed percolation clusters affect the characteristics of the directed polymer below the critical concentration.

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Several models have been developed for directed path in disordered media. Two well known models are directed polymer and directed percolation [1,2]. While the directed polymer is based on global optimization, the directed percolation can be described as a local process. The directed polymer is considered as a model for physical processes such as tearing or cracks [3], while the directed percolation has been used to model for example invasion of low viscosity fluid in high viscosity[4], as well as the interface of liquids in materials such as paper[5]. The relation between the two models is not yet clear and even controversial, as discussed below.

The two models of directed polymer and directed percolation can be described in a similar way as follows: In a square lattice which was cut along its diagonal and oriented as a triangle whose apex is up and the diagonal is its base, we assign to each bond a random number. Of all the paths leading from the apex (the origin) to the base we refer only to those whose direction is *always* down to the base. For each one of these paths we calculate the sum of the random numbers along it.

In the directed polymer model, the sum of the random numbers along the path defines its value. We focus our interest on the path of minimal value (which we call "the optimal path"), and define the roughness exponent ν by $D \sim t^\nu$, where D is the mean distance of the endpoint of the optimal path from the center of the base, t is the size of the triangle, and ν is the roughness exponent. In this model the random numbers are usually taken from a continuous distribution, and thus the optimal path is uniquely defined. Huse and Henley [1] who introduced this model found that $\nu \cong 2/3$ and the value $2/3$ is considered to be exact. Another exponent which characterizes this model is ω , defined by $\sigma_E \sim t^\omega$ where E is the value of the optimal path, and σ_E is its standard deviation. These two exponents are related by the scaling relation $\omega = 2\nu - 1$ [1].

In the directed percolation model the random numbers are taken from a bimodal (0,1) distribution, and we can define a percolation cluster as the collection of lattice sites which are connected by zero sum paths to the origin. In this model we can also define a roughness exponent by $W \sim L^\nu$ where W is the mean width and L is the mean length of a percolation cluster. W and L depend on the probability to get 0 (denoted by p) by the relations $W \sim (p_c - p)^{-\nu_\perp}$, $L \sim (p_c - p)^{-\nu_\parallel}$, where p_c is the critical probability $\simeq 0.6447$, ν_\perp is the transverse exponent $\simeq 1.097$, and ν_\parallel is the longitudinal exponent

$\simeq 1.733$ [4]. Since $\nu = \nu_{\perp}/\nu_{\parallel} \simeq 0.633$, there are two independent exponents in this model, compared to only one in the directed polymer model.

Huse and Henley [1] refer to the possibility that in the directed polymer model the random numbers are taken from a bimodal probability distribution, and stated that the roughness exponent has the same value as in the continuous case, i.e. $\nu = 2/3$. On the other hand, more recently, Lebedev and Zhang [6] claimed that in the bimodal case, for all $p \leq p_c$ the exponent is actually the same as in the directed percolation case, i.e. $\nu \simeq 0.633$.

In this Letter we address this controversial issue by studying the directed polymer model with bimodal (0,1) distribution. The numerical simulations lead to the following main conclusions:

1. For $p < p_c$ and for $p > p_c$, for large t the value of the roughness exponent is $\simeq 2/3$, the directed polymer value.
2. For $p = p_c$, the value of the roughness exponent is $\simeq 0.633$, the directed percolation value.
3. For $p < p_c$ and $t \gg L$, the sum of the random numbers along the optimal path is inversely proportional to the mean directed percolation cluster length, i.e. $E \sim t/L$.

It should be emphasized that a study of the directed polymer model yields dependence on both the critical probability and mean percolation cluster length, L , of directed percolation. So the general conclusion is that the two models are closely related.

In this study the random numbers are taken from a bimodal (0,1) distribution, and thus there is usually more than one optimal path and more than one optimal endpoint. If there exists a real percolation cluster spanning from the origin to the base, all the optimal endpoints (and no other point of the base) belong to this percolation cluster, but otherwise, the optimal paths and endpoints are not necessarily connected in any special manner. The set of optimal endpoints can be characterized by two variables:

1. D_0 - the distance of the center of the optimal endpoints set from the center of the base. The center of the set is computed by $(X_l + X_r)/2$ where X_l and X_r are the leftmost and rightmost points of the set.

2. W_0 - the width of the optimal endpoints set, which is computed by $W_0 = (X_r - X_l)$.

It is expected that for large t , $D_0 \sim t^{\nu_D}$, and $W_0 \sim t^{\nu_W}$, where ν_D and ν_W are in the range $[0,1]$. In order to test this hypothesis and estimate ν_D and ν_W , we define the local roughness exponent of a variable V at the point t , as the slope of the curve relating the logarithm of V to the logarithm of t . The numerical simulations provide estimates for the local exponents of D_0 and W_0 by computing $(\log V(2t) - \log V(t/2))/\log 4$, where V is either D_0 or W_0 . These estimates are presented in Figures 1 and 2 for 5 probabilities: $p = 0.25$, $p = 0.5$, $p = 0.62$, $p = 0.64$ and $p = 0.6447 (\simeq p_c)$. In Figure 1 it is seen that the slope of the $p \simeq p_c$ curve is stable at a value $\simeq 0.63$, while as p is further below p_c , the slope seems to crossover from $\simeq 0.63$ to a value close to $2/3$. In Figure 2 it is seen that the slope of the $p \simeq p_c$ curve converges to a value $\simeq 0.63$, while for $p < p_c$ the slope declines and approaches (for the probabilities further from p_c) a value close to $1/3$. These crossovers from directed percolation exponents to directed polymer exponents can be explained due to the relation between the triangle size t and the longitudinal correlation length $\xi_{||}$ [2]. For $t < \xi_{||}$ we expect properties of directed percolation, while for $t > \xi_{||}$ we expect properties of directed polymer.

The results presented in Figure 2 deserve some explanation, as one might expect that W_0 should be proportional to W - the width of a typical directed percolation cluster (in the relevant probability). If this was the case, we would get in Figure 2 an initial slope of 0.63, and then the slope would decline to zero. Evidently, the picture is different, as for each probability $p < p_c$ the slope will eventually converge to a value close to $1/3$. The reason for this lies in the ultrametric tree structure of the directed polymer problem. It was shown [7] that in the continuous distribution case, the region of endpoints of paths whose value difference from the value of the optimal path is smaller than a (small) constant, increases as $t^{1/3}$. In our case of discrete values, the same should hold for zero difference between the values of the optimal paths.

The main conclusion from Figures 1,2 is that for $p < p_c$ and sufficiently large t , the optimal endpoints set of width $\sim t^{1/3}$ is located around a point whose distance from the center is $\sim t^{2/3}$. As for $t \gg 1$, $t^{1/3}$ is negligible compared to $t^{2/3}$, the situation is similar to the continuous distribution case, where instead of one endpoint there is a "cloud" of endpoints whose distance from the center is $\sim t^{2/3}$.

For $p > p_c$ there is a finite probability for points of the base to belong to a real directed percolation cluster, so that $W_0 \sim t$. On the other hand, we find numerically that $D_0 \sim t^{1/2}$. As $W_0 \gg D_0$, it is certain that members of the optimal endpoints set are found in both sides of the center, and another definition of the distance is needed. A definition that takes into account the fact that the directed polymer is equally likely to choose any one of the optimal paths, is $D_r = \sum_i n_i |x_i| / \sum_i n_i$, where n_i is the number of optimal paths whose endpoint is x_i . The local exponents of D_r are presented in Figure 3 for 3 probabilities: $p = 0.5$, $p = 0.6447 (\simeq p_c)$, and $p = 0.75$.

As might be expected, the results for $p < p_c$ and $p \simeq p_c$ approach the values $2/3$ and 0.63 respectively, further supporting that for $p < p_c$, $\nu = 2/3$, while for $p = p_c$, $\nu \simeq 0.63$. The results for $p > p_c$ are inconclusive, as the local exponent does not yet converge at this triangle size. But, for $p > p_c$ it is expected that the situation is governed by one big percolation cluster. If this is the case, it is possible (and much less computer power demanding) to "grow" percolation clusters and to find the dependence of D_r on t in that way. The local exponents of D_r for directed polymer at $p = 0.75$ and for directed percolation at the same probability are shown in figure 4.

As can be seen in figure 4, the results of directed polymer and directed percolation are statistically indistinguishable, and it is quite safe to assume that this situation will not change for larger t . The local exponent for directed percolation approaches a value $\simeq 2/3$, a result which was obtained earlier by Balents and Kardar [8]. Thus we conclude that for $p > p_c$, ν of directed polymer has the same value as for $p < p_c$, i.e. $\nu \simeq 2/3$.

It was mentioned above that the results of Figures 1,2 can be explained in terms of the longitudinal correlation length $\xi_{||}$. More direct result follows from a picture of the optimal path as a series of zero sum segments whose mean length is L , connected by single bonds of value 1. (Obviously, this picture holds only for $t \gg L$). According to this picture, we expect that $E \sim t/L$, and for fixed t , $E \times L \sim \text{constant}$. Figure 5 presents the results for E/E_0 , L/L_0 , and $(E \times L)/(E_0 \times L_0)$ in the range of probabilities $0.5 < p < 0.64$, where E_0 is E at $p = 0.5$ and L_0 is L at $p = 0.5$.

As can be seen in Figure 5, E and L form mirror reflection of each other over two orders of magnitude, while $E \times L$ is a slightly decreasing function of p . So there is almost one to one correspondence between results obtained from the directed polymer model (the values of E), and results obtained *independently* from the directed percolation model (the values of L).

In conclusion, it was shown that the bimodal distribution directed polymer model can be characterized in terms of the directed percolation model, and that the percolation threshold probability p_c plays a critical role in the directed polymer case.

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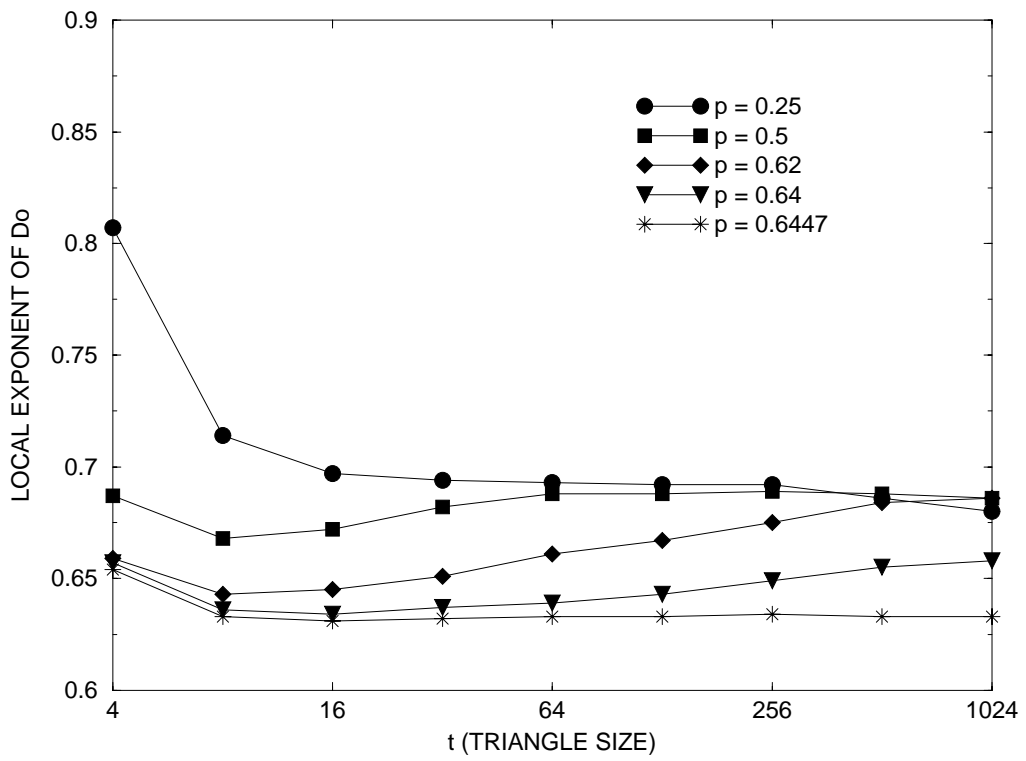


Figure 1: Plot of the local exponent of D_0 as a function of t for several values of p .

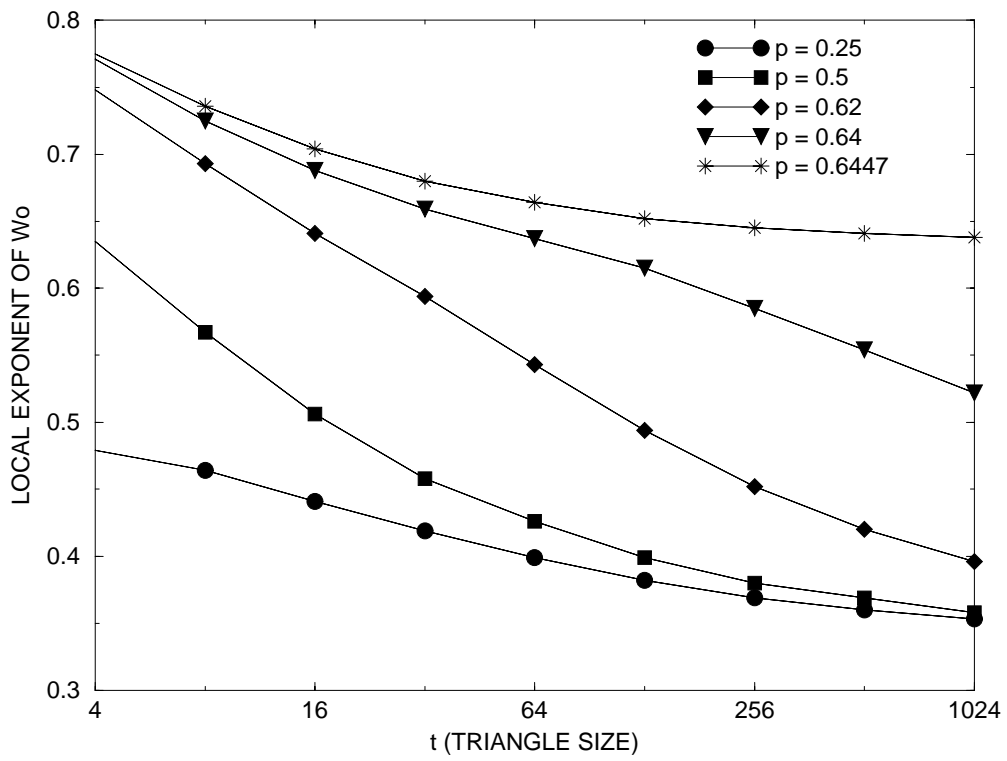


Figure 2: Plot of the local exponent of W_0 as a function of t for several values of p .

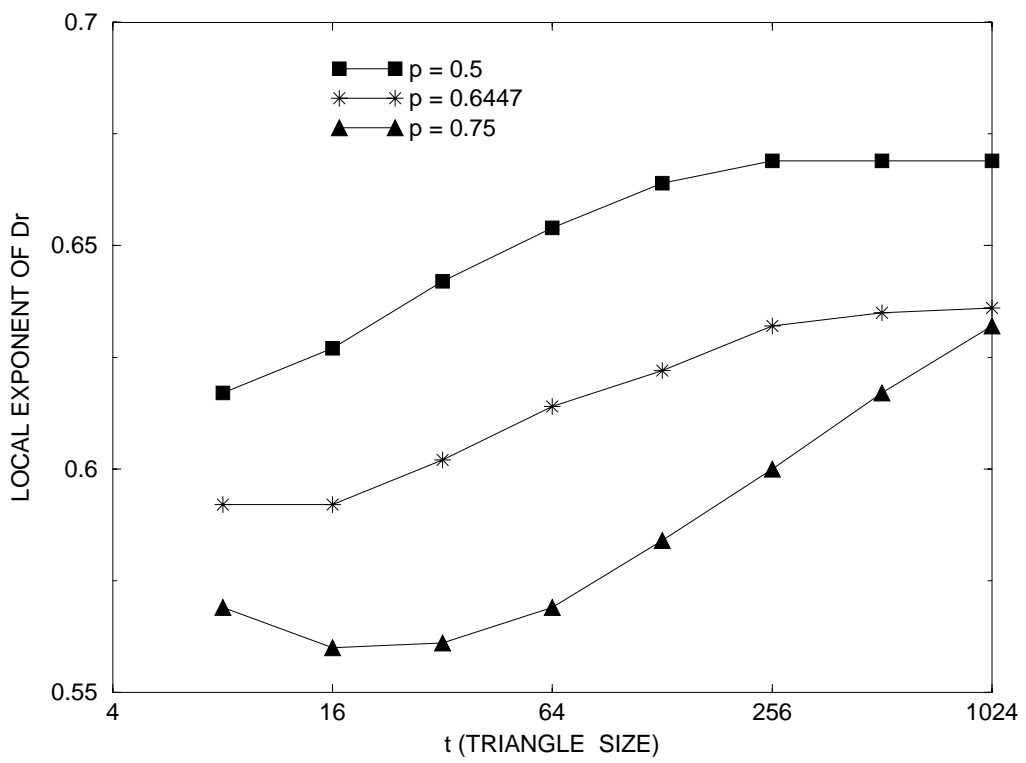


Figure 3: Plot of the local exponent of D_r as a function of t for several values of p .

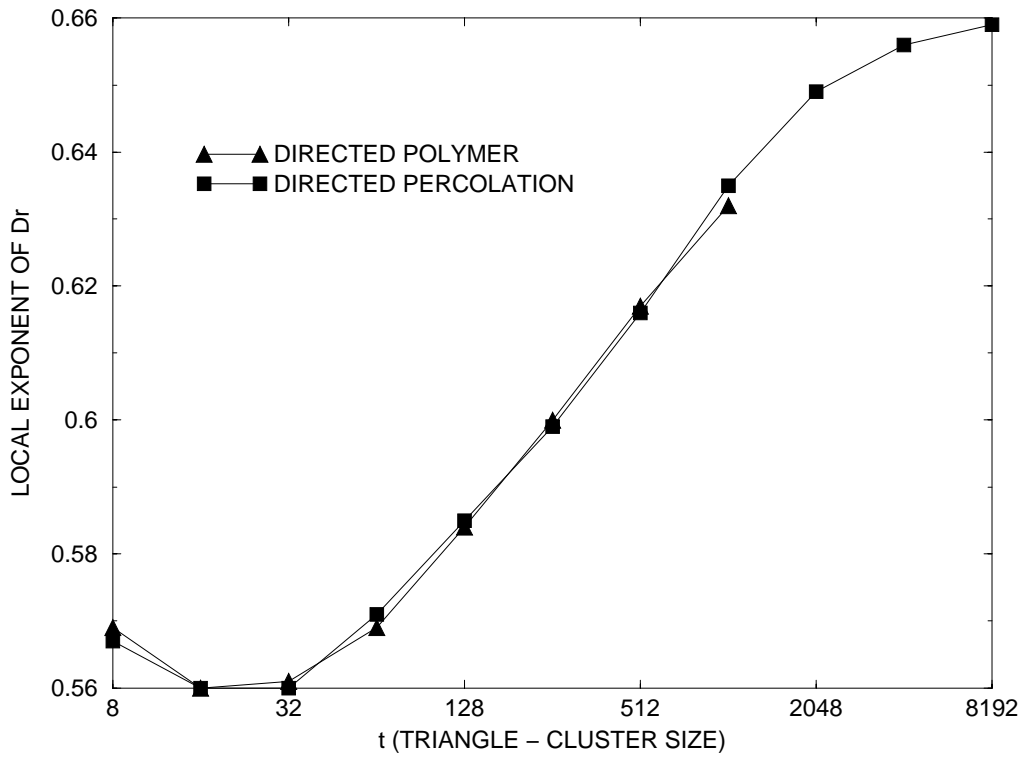


Figure 4: Plot of the local exponent of D_r as a function of t for directed polymer and directed percolation at $p = 0.75$.

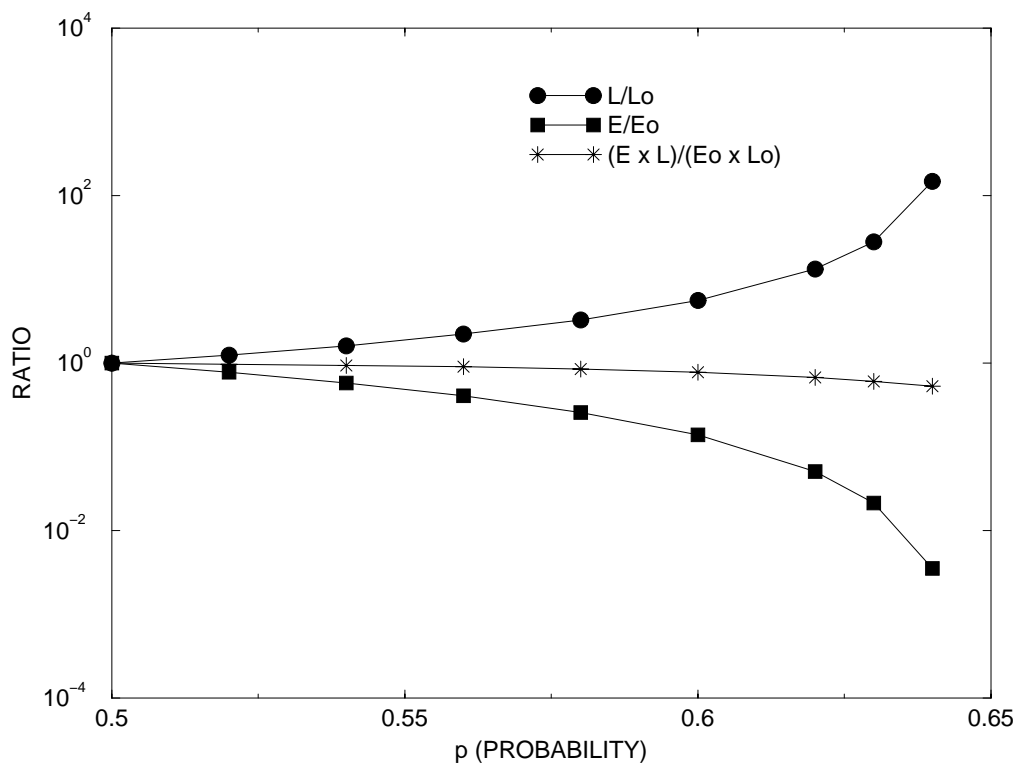


Figure 5: Plot of E/E_0 , L/L_0 , and $(E \times L)/(E_0 \times L_0)$ as a function of the probability p . E_0 and L_0 are the values at $p = 0.5$.