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Directed polymer – directed percolation transition: the strong disorder case

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Abstract. The transition of physical properties in disordered systems from strong disorder characteristics to weak disorder characteristics is studied for the directed polymer case. It is shown analytically that this transition is governed by the ratio $\rho(p_c)/k$, where ρ is the probability density of the maximal bond of the optimal Min-Max path, p_c is the critical probability of directed percolation, and k is the degree of disorder. This analytic result is found to be in agreement with numerical results related to this transition.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 05.50.+q Lattice theory and statistics (Ising, Potts, etc.)

Many physical properties in disordered systems such as transport, polymers' structure, epidemics, or optimal paths [1–4], depend on the type of disorder. In particular, two limits of disorder have been studied extensively, strong and weak disorder which yield usually different universality classes. For example, while the length of optimal paths between two sites represented by a distance r in weak disorder yields for all lattices in d dimensions $l \sim r$ [5], in strong disorder $l \sim r^{\alpha}$, where $\alpha \simeq 1.22$ in $\mathrm{d}=2$, and $\alpha \simeq 1.42$ in $\mathrm{d}=3$ [6,7]. Moreover, while in weak disorder there is no single bond (or site) whose removal is associated with a significant change in the physical properties of the system, in the strong disorder case such a bond (or site) always exists [1].

Although the transition between weak and strong disorder was studied numerically [4,8], the scaling of the transition was not explained, and no analytical derivation of the transition properties has been presented. This study is concerned with the transition in the directed polymer case, and an analytical derivation of the dependence of this transition on the system size and on the degree of disorder is presented below. The analytical results are found to be in agreement with numerical results obtained for this transition. Though the derivation is performed for the specific case of directed optimal paths, the simplicity of its conclusion suggests that similar analysis can be applied to the regular (non-directed) lattice.

The manuscript begins with a description of the directed polymer model [9] with strong disorder, and the similarity with the directed percolation model [10,11] is explained. Afterwards, numerical results related to the transition from strong disorder to weak disorder are presented. Simple theoretical arguments which explain these numerical results are presented at the end.

There are many ways to realize strong disorder in the directed polymer model. In the simplest one, which was utilized in the present numerical study, the bonds of the lattice are assigned with random numbers b_i taken from uniform distribution in the range (0,1). The energy value of the bond is defined as $B_i = e^{kb_i}$, where k is a constant representing the degree of disorder. The larger is k, the broader is the disorder. In the extreme case, the energy value of any path of length t is determined by the energy value of only one bond, and this situation is similar to the case in which the value of a path is defined as the value of b_{max}, where b_{max} is the maximal bond value along that path. (Note that since k is positive, $B_{max} = e^{kb_{max}}$, and the maximal bond value criterion might be applied to either b or B.) The optimal path in this case is the one with the lowest value of b_{max}, and thus a Min-Max criterion is applied to the choice of the optimal path [12,13].

Suppose that in the same random lattice, the following transformation is applied: $B_i^* = 0$ if $b_i \leq b_{max}$; $B_i^* = 1$ if $b_i > b_{max}$. The Min-Max optimal path is transformed to a path of zero valued bonds B_i^*, and is thus a part of a directed percolation cluster of length t. Thus, in the strong disorder limit, the optimal path is identical to the one obtained using the Min-Max criterion, and this path has characteristics similar to those of directed percolation clusters. In particular, the roughness exponent ν has the

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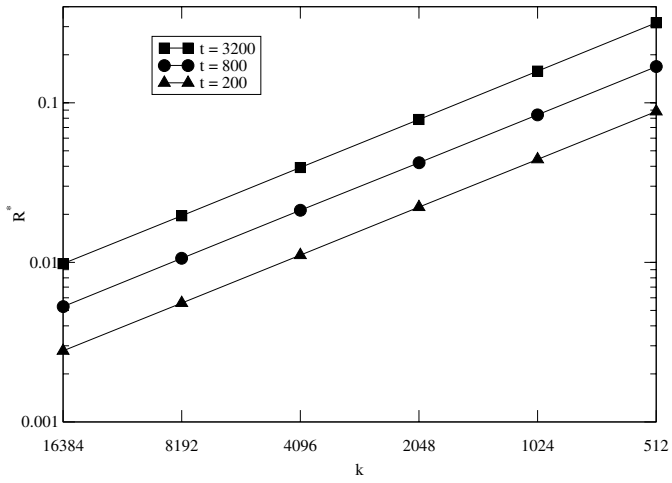

Fig. 1. The values of R^* as a function of $1/k$ for $t = 3200$ (upper curve), $t = 800$ (middle curve), and $t = 200$ (lower curve).

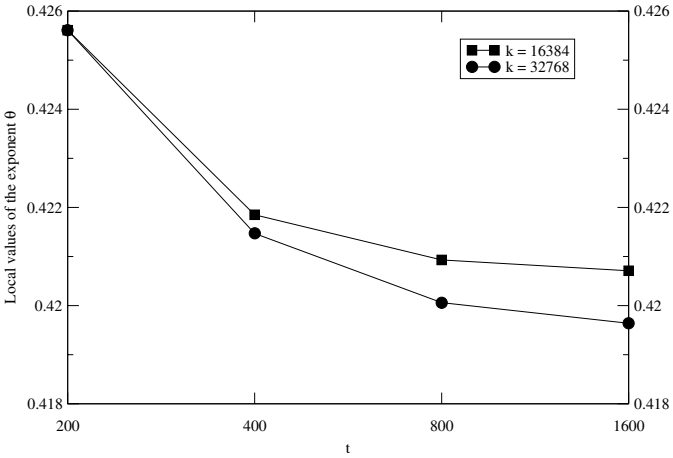

Fig. 2. The local values of the exponent θ computed from the difference series $\Delta R^*(t)$ for $k = 16384$ (squares), and $k = 32768$ (circles).

directed percolation value of $\simeq$0.63, and not the directed polymer value of 2/3 [13].

For any value of k, the ratio between the sum of bonds' values $\sum B_i$, and the maximal bond value B_{max}, increases as t increases. Note that when $R(k,t) \equiv \sum B_i/B_{max}$ is of the order of 1, the strong disorder limit is valid, while when $R(k,t) \gg 1$, the system is in the weak disorder limit. In the following, this transition is studied both numerically and analytically.

The degree of deviation from the strong disorder limit of 1 can be measured by $R^*(k,t) \equiv \log R(k,t)$. Figure 1 presents on a log-log scale the dependence of R^* on $1/k$ for three values of t: $t = 200$; $t = 800$; $t = 3200$. In the three cases, the curves are nearly straight lines whose slope is 0.999(4), and the relation $R^* \sim 1/k$ is quite well established by the numerical data.

The relation between R^* and t is not so simple, and the numerical results suggest the relation $R^* \sim t^\theta$. In order to estimate the exponent θ, Figure 2 presents the local values of θ computed by $\log_4(\Delta R^*(2t)/\Delta R^*(t/2))$, where the difference series $\Delta R^*(t)$ is defined by: $\Delta R^*(t) \equiv R^*(t) - R^*(t/2)$. It is easy to confirm that if $R^*(t) \sim t^\theta$, $\Delta R^*(t)$ should have similar dependence on t, while convergence to the asymptotic value of the exponent is usually faster for the data computed from the difference series. These local values of θ are presented in Figure 2 as a function of t for the cases $k = 16384$ and $k = 32768$, and the two sets of estimates can be summarized as follows:

1. The local values estimated for $k = 32768$ are lower than those estimated for $k = 16384$, and a detailed analysis of the data showed that this difference is statistically significant.
2. The local values of the exponent θ in both data sets are still decreasing, and the values estimated for $k = 32768$ decrease towards a lower value than those estimated for $k = 16384$.
3. The above observations lead to the conclusion that the asymptotic k curve should be lower than the one estimated for $k = 32768$, and that the asymptotic value of θ is lower than the last data point of the lower curve, which is $\simeq$0.420.

Though the data supplies only a solid upper bound on the value of θ, it is not necessary to guess a value for the lower bound, since the theoretical analysis presented below yields a value which is very close to 0.420.

Intuitively, these results can be explained with the aid of the former results (and approach) presented in [12], and familiarity with the concept of self organized critical phenomena introduced in that study might help to follow the line of reasoning presented in the following: The transition should be determined by the probability that B_{max} is not the sole dominant factor in the sum $\sum B_i$. This would happen when the second highest valued bond, B_2, is not much smaller than B_{max}. The requirement: $B_{max}/B_2 < M$, where M is some constant (say 10), leads to the requirement: $b_{max} - b_2 < \log M/k$, from which follows the $1/k$ dependence shown in Figure 1. The dependence of $\Delta b \equiv b_{max} - b_2$ on the path length t is determined by the probability densities of b_{max} and b_2. Since b_2 and b_{max} are the two highest valued bonds of the optimal path, their probability densities should be similar. It was shown in [12] that as t increases, the probability density of b_{max} approaches the form of a δ function centered around p_c, the critical probability of directed percolation. In the vicinity of p_c, the probability density of b_{max} (and hence of b_2) grows in proportion to t^θ; $\theta \simeq 0.417$ [12,14]. Since b_{max} is always higher than b_2, it is natural to guess that in the same way that b_{max} gets closer to p_c, b_2 gets closer to b_{max}, with the same t dependence of $\theta \simeq 0.417$, in accordance with the numerical results presented in Figure 2. A more formal derivation of this argument is presented below.

The first step of the derivation is already described above: Since the sum of the bonds' values in the extreme case is determined by B_{max}, the ratio between the sum and B_{max} is determined by B_2, where B_2 is

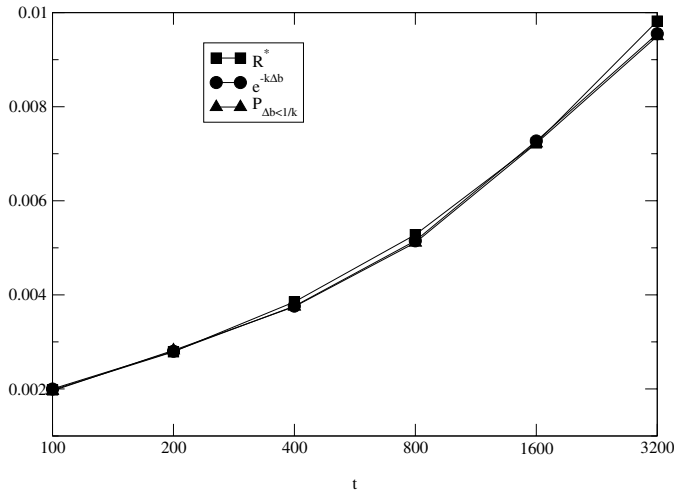

Fig. 3. A comparison between the values of R^* (squares), $\overline{e^{-k\Delta b}}$ (circles), and $P_{\Delta b<1/k}$ (triangles). The three variables are computed from the numerical data obtained for $k = 16384$.

the second highest valued bond along the optimal path: $\sum B_i/B_{max} \simeq (e^{kb_{max}} + e^{kb_2})/e^{kb_{max}} = 1 + e^{-k\Delta b}$, where $\Delta b \equiv b_{max} - b_2$. Since $\overline{e^{-k\Delta b}} \ll 1$ in the strong disorder case, the following relation should hold:
$R^*(k,t) \equiv \log R(k,t) \simeq \overline{e^{-k\Delta b}}(t) = \int P(\Delta b, t) e^{-k\Delta b} d\Delta b$, where $P(\Delta b, t)$ is the probability density of Δb in an optimal path of length t. Note that the integration in the last expression is performed only for positive values of Δb. Since Δb is positive, $e^{-k\Delta b}$ is a decreasing function of Δb, with significant values only in the $1/k$ vicinity of 0. The second variable in the integrand, $P(\Delta b, t)$, should follow the right hand side of a Gaussian distribution, which is also a decreasing function of Δb. For high enough values of k, the value of $P(\Delta b, t)$ in the $1/k$ vicinity of $\Delta b = 0$ is $\simeq P(0,t)$, and thus: $R^*(k,t) \simeq P(0,t) \int e^{-k\Delta b} d\Delta b = P(0,t)/k$. From this relation follows the $1/k$ dependence of $R^*(k,t)$ demonstrated in Figure 1.

Note that $P(0,t)/k$ is simply the probability of Δb to be smaller than $1/k$, denoted by $P_{\Delta b<1/k}(t)$. In the numerical study, the value of Δb was also computed for each optimal path, and in order to check the validity of the presumed relation: $R^*(k,t) \simeq \overline{e^{-k\Delta b}}(t) \simeq P_{\Delta b<1/k}(t)$, the data related to these three variables is presented in Figure 3 as a function of t for $k = 16384$. As can be seen, the three curves are almost indistinguishable.

Since it is shown above that $R^*(k,t) \simeq P(0,t)/k$, the t dependence of $R^*(k,t)$ follows the t dependence of $P(0,t)$, i.e., the t dependence of the probability that b_2 is very close to b_{max}. In order to find the t dependence of $P(0,t)$, note that if the transformation:
$B_i^* = 0$ if $b_i \leq p$; $B_i^* = 1$ if $b_i > p$ is applied to a random lattice, then the value $p = b_{max}$ is the lowest value which yields a directed percolation cluster of zero valued bonds B_i^*. Consider N such random lattices, and suppose there are n random lattices whose b_{max} is lower than some constant p. If the same transformation is applied to all the random lattices, the n lattices have t length directed percolation clusters (of zero valued bonds B_i^*), and thus $n/N \simeq S(p,t)$, where $S(p,t)$ is the probability to have a directed percolation cluster of length $\geq t$ in a lattice whose bonds have the probability p to be zero valued. (Note that b is taken from a uniform distribution in the range (0,1), and thus the probability that $B_i^* = 0$ is simply p.) From the above discussion follows that $S(p,t) = \int^p \rho(b_{max}, t) db_{max}$, and thus that $\rho(p,t) = dS(p,t)/dp$, where ρ denotes the probability density (or density of states), and p denotes the probability to have a zero valued bond. (p and b_{max} have interchangable roles in the notation, bearing in mind the transformation described above.)

Consider now the second highest bond b_2. As much as b_{max} can support a directed percolation cluster of length t, b_2 can support a cluster of comparable (though smaller) length, and thus, the probability density of b_2 is similar to the probability density of b_{max}. However, in any specific random lattice, it is known that $b_2 < b_{max}$, and thus, the conditional probability $\rho(b_2|b_{max})$ should be used. Assuming that apart from the fact that $b_2 < b_{max}$, there is no correlation between b_2 and b_{max}, the following relation should hold: $\rho(b_2|b_{max}) \simeq \rho(b_2)/S(b_{max})$. Obviously, when $\Delta b = 0$, $b_2 = b_{max}$, and from the above relation follows:
$P(0,t) = \int \rho(b_{max}, t) * \rho(b_2 = b_{max}|b_{max}, t) db_{max} \simeq \int \rho(b_{max}, t) * \rho(b_{max}, t)/S(b_{max}, t) db_{max}$ where the last expression utilizes the fact that $\rho(b_2, t) \simeq \rho(b_{max}, t)$.

The probability density $\rho(b_{max}, t)$ was studied in [12], where it was shown that significant values of $\rho(b_{max}, t)$ are found only in the $t^{-1/\nu_\parallel}$ vicinity of p_c, while in that vicinity the value of $\rho(b_{max}, t)$ is $\sim t^{1/\nu_\parallel - \delta}$. $\nu_\parallel$ is the longitudinal correlation exponent of directed percolation, defined by $\xi \sim (p_c - p)^{-\nu_\parallel}$, and its value in the two dimensional case is $\simeq$1.734 [14]. The decay exponent δ is defined by $S(p_c, t) \sim t^{-\delta}$, and its value is $\simeq$0.1595 [14]. Since $S(b_{max}, t)$ at p_c is $\sim t^{-\delta}$, the value of the above integral is $\sim t^{1/\nu_\parallel - \delta}$, which is $\sim \rho(p_c, t)$. The final result is thus very simple: $P(0,t) \sim \rho(p_c, t)$, where $P(0,t)$ is the probability that $b_2 = b_{max}$, and $\rho(p_c, t)$ is the probability that $b_{max} = p_c$. The resultant estimate for the growth exponent is $\simeq$0.417, in agreement with the data presented in Figure 2.

In conclusion, it is shown that the transition from strong disorder characteristics to weak disorder characteristics is governed by the ratio $\rho(p_c, t)/k$. where $\rho(p_c, t)$ is the probability density of b_{max} in the vicinity of the critical probability of directed percolation: $\rho(p_c, t) \sim t^\theta$. This analytical result is found to be in agreement with numerical results related to this transition, and its simplicity suggests that similar relations might be found in other cases of transitions from strong disorder to weak disorder.

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