EUROPHYSICS LETTERS 15 January 2006
Europhys. Lett., **73** (2), pp. 178–182 (2006)
DOI: 10.1209/epl/i2005-10382-9

Optimal paths as correlated random walks

E. Perlsman and S. Havlin
Department of Physics, Bar-Ilan University - Ramat-Gan 52900, Israel

received 31 July 2005; accepted in final form 22 November 2005
published online 9 December 2005

PACS. 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion.
PACS. 05.50.+q – Lattice theory and statistics (Ising, Potts, etc.).

Abstract. – A numerical study of optimal paths in the directed polymer model shows that the paths are similar to correlated random walks. It is shown that when a directed optimal path of length t is divided into 3 segments whose length is $t/3$, the correlation between the transversal movements along the first and last path segments is independent of the path length t. It is also shown that the transversal correlations along optimal paths decrease as the paths approach their endpoints. The numerical results obtained for optimal paths in 1+4 dimensions are qualitatively similar to those obtained for optimal paths in lower dimensions, and the data supplies a strong numerical indication that 1+4 is not the upper critical dimension of this model, and of the associated KPZ equation.

The directed polymer model [1] is a well-studied [2] model in the field of disordered systems. The model is concerned with directed optimal paths in random media, which are characterized by two growth rate exponents, ω and ν. The energy exponent ω determines the energy variability of a path of length t by the relation $\Delta E \sim t^{\omega}$, and the roughness exponent ν determines the mean transversal distance of the optimal paths from the origin, through the relation $D \sim t^{\nu}$. These two exponents are connected by the Huse-Henley scaling relation: $\omega = 2\nu - 1$ [3]. An open and controversial question is whether this model (and the associated KPZ equation [4]) has an upper critical dimension (UCD), above which the optimal paths are uncorrelated and thus $\nu = 0.5$ (and $\omega = 0$). On one side of the controversy stand theoretical studies which suggest the existence of an UCD, usually ≤ 4 [5–10]. On the other side stand both theoretical studies which suggest no finite UCD [11–13], together with numerical studies which find that in the 4-dimensional case the value of $\chi \equiv \omega/\nu$ is > 0 [14, 15], and that ν in the 1+4 dimensional case is > 0.5 [16]. (The exponent χ is estimated for rough surfaces which are characterized by the KPZ equation). However, these numerical results fail to convince the proponents of the UCD hypothesis, which claim that the variables whose rates of growth are measured in these numerical studies, are too close to one lattice unit [17, 18].

The present work studies numerically the correlations between the transversal movements along the optimal paths. It is shown that when a directed optimal path of length t is divided into 3 segments whose length is $t/3$, the correlation between the transversal movements along the first and last path segments is independent of the path length t. It is also shown that the transversal correlations along optimal paths decrease as the paths approach their endpoints. The numerical results obtained for optimal paths in 1+4 dimensions are qualitatively similar to

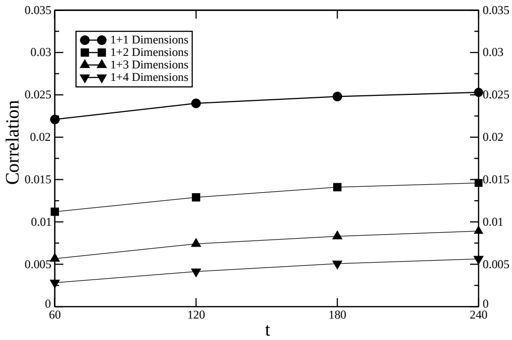

Fig. 1 – The correlations between the first and third path segments of length $t/3$, in optimal paths of 1+1, 1+2, 1+3 and 1+4 dimensions.

those obtained for optimal paths in lower dimensions, and the data supplies a strong numerical indication that 1+4 is not the upper critical dimension of this model.

In order to study the correlations, a simple criterion is used. The optimal path of length t is divided into 3 path segments of length $t/3$, and the direction of the transversal movement of each path segment is recorded. Denoting by $+1$ movement in the positive direction of the transversal coordinate, and by -1 movement in the negative direction, the correlation between two path segments is measured by the mean value of the product of their movements. Note that in some cases the movement is nil, and thus even the correlation of the path segment with itself is < 1, but the probability of these cases decreases towards 0 with the length of the path segments. Note also that the correlation measured in this way is simply the difference between the probability that the movement along the two path segments is in the same transversal direction, and the probability that the movement is in opposite transversal directions.

Figure 1 presents the correlations between the first and last path segments of paths which are divided into 3 segments of length $t/3$, for the 1+1, 1+2, 1+3 and 1+4 dimensional cases. Naturally, the strength of the correlations decreases with the dimension, but the important fact is that in all cases the correlations approach constant values which are independent of t. Note that the lower is the curve, the faster it rises towards its asymptotic value: The ratios between the correlations at $t = 240$ and at $t = 60$ are $\simeq 1.14$, 1.30, 1.58 and 2.00 for the 1+1, 1+2, 1+3 and 1+4 dimensional cases. Note also that no sign of any exceptional behaviour can be observed in the data related to the 1+4 dimensional case.

These constant correlations between the path segments can only be a consequence of positive correlation between the transversal directions of each pair of steps along the optimal path, and it is intuitively clear that such correlations should also affect the value of ν. Figure 2 validates this intuition for the case of long-range–correlated random walks (LRCRW). In each step of the LRCRW, the walker advances one step forward, and at the same time one step to the right or to the left. The correlation between the transversal steps is of the type $c(t) \sim t^{-\delta}$, where t is the longitudinal distance between the steps. For $\delta < 1$ the roughness exponent of the LRCRW has the value $\nu = 1 - \delta/2$, while for $\delta \geq 1$, $\nu = 0.5$.

The figure presents the correlations between the first and last path segments of paths which are divided into 3 segments of length $t/3$, for LRCRW whose values of δ are 0.90, 0.95 and 1. The correlations in the first two cases rise towards asymptotic values which are independent of t, but do depend on δ, while in the case of $\delta = 1$, the correlations descend towards an asymptotic value which, as shown in the following, should be 0. Thus, constant correlations

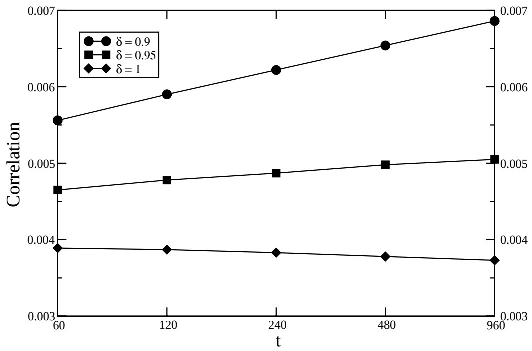

Fig. 2 – The correlations between the first and third path segments of length $t/3$, in LRCRW whose values of δ are 0.9, 0.95 and 1.

between path segments whose length and distance are $\sim t$ characterize LRCRW whose $\delta < 1$ and $\nu > 0.5$. In view of the picture presented in fig. 1, note that the data presented in fig. 2 does distinguish between the cases in which $\nu > 0.5$, and the case in which $\nu = 0.5$.

Consider a path (or walk) of length $2t$, and divide it into two segments of length t. Denote by $\Delta x_f(t)$ and $\Delta x_s(t)$ the transversal displacements of the first and second path segments, and by $\Delta x(2t)$ the transversal displacement of the whole path. Note that $\Delta x(2t) = \Delta x_f(t) + \Delta x_s(t)$, and that the mean transversal distance between the origin and endpoint of a path of length t is $d(t) \equiv E(\sqrt{\Delta x^2(t)})$, where E denotes the mean value.

A basic relation in probability is

$$Var(\Delta x(2t)) = Var(\Delta x_f(t)) + Var(\Delta x_s(t)) + 2Cov(\Delta x_f(t), \Delta x_s(t))$$

where Var and Cov are the variance and covariance of the variables in parentheses. The value of the exponent ν is defined by the asymptotic value of $\log_4(Var(\Delta x(2t))/Var(\Delta x(t))) \equiv \log_4(R_1 + 2R_2)$, where the values of R_1 and R_2 are defined by

$$R_1 \equiv (Var(\Delta x_f(t) + Var(\Delta x_s(t)))/Var(\Delta x(t)),$$

$$R_2 \equiv Cov(\Delta x_f(t), \Delta x_s(t))/Var(\Delta x(t)).$$

In LRCRW, $Var(\Delta x_f(t)) = Var(\Delta x_s(t)) = Var(\Delta x(t))$, and thus $R_1 = 2$. The value of ν in this case is the asymptotic value of $0.5 + \log_4(1 + Cov(\Delta x_f(t), \Delta x_s(t))/Var(\Delta x(t)))$. The second factor in the parenthesis is the correlation coefficient between $\Delta x_f(t)$ and $\Delta x_s(t)$, which is closely related to the correlation as defined above and computed for figs. 1, 2. Thus, for LRCRW whose $\delta \geq 1$ and $\nu = 0.5$, the asymptotic value of the correlation, computed in either way, should be nil.

Looking back at fig. 1, there is no doubt that the asymptotic value of the correlations computed for the 1+4 dimensional case is > 0, and thus, as was verified in the numerical study, that R_2 is also > 0. If 1+4 is the UCD of the directed polymer model, then R_1 computed for this case should be < 2. In the following it is shown that the opposite is true.

The basic difference between LRCRW and directed optimal paths is that in LRCRW, the first t steps are not affected by the extension of the walk to length of (say) $2t$, while if a random lattice of length t is extended to length $2t$, the t steps of the (old) optimal path of length t are usually different from the first t steps of the (new) optimal path of length $2t$. As a consequence, in directed optimal paths, the variances of $\Delta x_f(t)$, $\Delta x_s(t)$

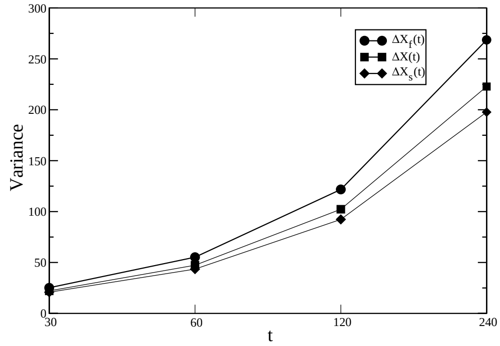

Fig. 3 – The variances of the variables $\Delta x_f(t), \Delta x(t)$ and $\Delta x_s(t)$. Each data point of the upper and lower curves is computed from optimal paths of length $2t$, while each data point of the middle curve is computed from optimal paths of length t.

and $\Delta x(t)$ are not necessarily equal, and as shown in fig. 3 for the 1+4 dimensional case, $Var(\Delta x_f(t)) > Var(\Delta x(t)) > Var(\Delta x_s(t))$. This result implies that the strength of the correlations between pairs of steps separated by the same longitudinal distance is a decreasing function of the longitudinal distance from the origin. Further numerical results indicate that in directed optimal paths of length t, the correlation between pairs of steps whose longitudinal distances from the origins are t^*, $t^* + T$, is $A(t^*/t)f(T)$, where for $1 \ll T \ll t$, $f(T) \sim T^{-\delta}$, and $A(t^*/t)$ is a decreasing function of its argument.

The decrease in the strength of the correlations along the optimal path is an outcome of the fact that the origin is fixed, while the endpoint is freely chosen from all the lattice sites at longitudinal distance t from the origin. In crude terms it might be said that the long-sighted walker locates a low-energy region in the bulk of the lattice, and starts moving in its direction. Thus, the direction of the path is more clearly defined near its start than near its end.

In cases that the correlations between path segments are positive, $R_2 > 0$, and a lower bound on the value of ν is $\log_4 R_1$. If $R_1 > 2$, $\nu > 0.5$. As noted above, in LRCRW $R_1 = 2$, while, as shown in fig. 3, this equality does not necessarily hold for directed optimal paths. Figure 4 presents the estimated values of R_1 for the 1+1 and 1+4 dimensional cases. In the 1+1 dimensional case $R_1 \simeq 2.22$, and the lower bound on the value of ν is 0.57. In the 1+4

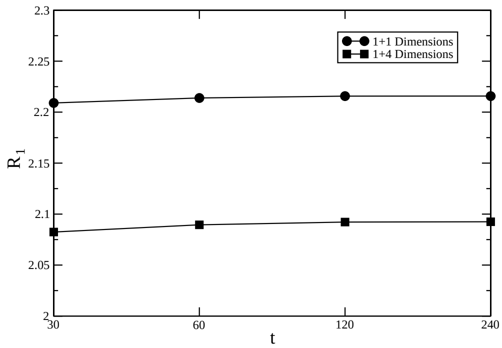

Fig. 4 – The value of R_1 in optimal paths of 1+1 and 1+4 dimensions. Note that the lower curve is the sum of the upper and lower curves of fig. 3, divided by the middle curve of that figure.

dimensional case $R_1 \simeq 2.09$, and the lower bound on the value of ν is 0.53. This data supplies a direct numerical evidence that 1+4 is not the UCD of the directed polymer model.

In summary, this numerical study leads to two main conclusions: The first is that the correlations between path segments of directed optimal paths are similar to those of LRCRW whose values of ν are > 0.5. The second conclusion is that unlike LRCRW, the correlations between steps of optimal paths are a decreasing function of the longitudinal distance from the origin. No sign for the existence of an UCD ≤ 4 can be observed in the data, and direct numerical evidence presented at the end confirms that the value of ν in the 1+4 dimensional case is > 0.5.

* * *

We thank the Israel Science Foundation and the European NEST Project DYSONET for financial support.

REFERENCES

[1] Kardar M. and Zhang Y.-C., *Phys. Rev. Lett.*, **58** (1987) 2087.
[2] Halpin-Healy T. and Zhang Y.-C., *Phys. Rep.*, **254** (1995) 215.
[3] Huse D. A. and Henley C. L., *Phys. Rev. Lett.*, **54** (1985) 2708.
[4] Kardar M., Parisi G. and Zhang Y.-C., *Phys. Rev. Lett.*, **56** (1986) 889.
[5] Halpin-Healy T., *Phys. Rev. A*, **42** (1990) 711.
[6] Schwartz M. and Edwards S. F., *Europhys. Lett.*, **20** (1992) 310.
[7] Bouchaud J. P. and Cates M. E., *Phys. Rev. E*, **47** (1993) R1455.
[8] Lassig M. and Kinzelbach H., *Phys. Rev. Lett.*, **78** (1997) 993.
[9] Katzav E. and Schwartz M., *Physica A*, **309** (2002) 69.
[10] Fogedby H. C., *Phys. Rev. Lett.*, **94** (2005) 195702.
[11] Tu Y., *Phys. Rev. Lett.*, **73** (1994) 3109.
[12] Castellano C., Gabrielli A., Marsily M., Munoz M. A. and Pietronero L., *Phys. Rev. E*, **58** (1998) R5209.
[13] Fogedby H. C., *Physica A*, **314** (2002) 182.
[14] Ala-Nissila T., *Phys. Rev. Lett.*, **80** (1998) 887.
[15] Marinari E., Pagnani A. and Parisi G., *J. Phys. A*, **33** (2000) 8181.
[16] Kim J. M., *Physica A*, **270** (1999) 335.
[17] Lassig M. and Kinzelbach H., *Phys. Rev. Lett.*, **80** (1998) 889.
[18] Colaiori F. and Moore M. A., *Phys. Rev. Lett.*, **86** (2001) 3946.