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"Logistic map": an analytical solution

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Abstract

An analytical solution for the well-known quadratic recursion, the *logistic map*, is presented. Our derivation is based on the analogy between this recursion and a probabilistic problem that can be solved analytically. The solution is represented as a power of a transfer matrix. The proposed method allows to solve a more general quadratic mapping.

Being one of the famous equations in physics and biology the logistic map needs no special introduction (a good review of the problem and relevant references can be found in [1,2]). In this paper we present an analytical solution for the logistic map recursion

$$f_{n+1} = \lambda f_n(1 - f_n), \qquad \text{with } f_0 \equiv f. \tag{1}$$

We begin with the following auxiliary probabilistic problem. Let a given particle at each time step give birth to another identical one with probability p or just survive with probability q $(q = 1 - p)$. The quantity under question is the probability $P_{n,k}$ to find an (n, k)-state, i.e., k particles after n time steps. It can be derived in two ways.

The first one is to write down a relation for $P_{n,k}$,

$$P_{n+1,k} = qP_{n,k} + p \sum_m P_{n,m}P_{n,k-m}. \tag{2}$$

This formula can be understood by the schematic diagram presented in Fig. 1. By switching to a generating function $\tilde{P}_n(x) = \sum_k P_{n,k}x^k$, Eq. (2) transforms to

$$\tilde{P}_{n+1}(x) = q\tilde{P}_n(x) + p[\tilde{P}_n(x)]^2. \tag{3}$$

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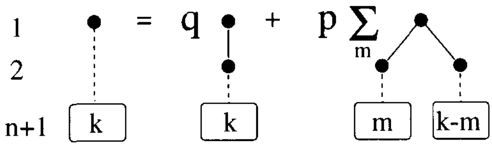

Fig. 1. Contribution of the first step to the *branching process* [3]. The dashed lines ended by a rectangular represent the probability for a given *generation* (here it is n or $n+1$) to have a *size* shown inside the rectangular.

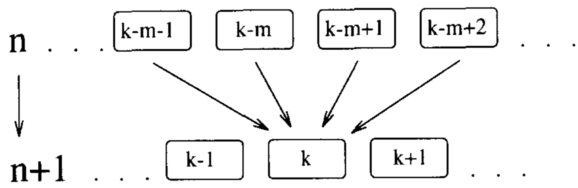

Fig. 2. Transition from an (n,k)-state to an $(n+1,k)$ one. There are $\binom{k-m}{m}$ possibilities to get k particles from $k-m$ ones; each possibility has a weight $p^m q^{k-2m}$.

The second way to calculate $P_{n,k}$ is illustrated in Fig. 2. It is straightforward to see that

$$P_{n+1,k} = \sum_{m=0}^{[k/2]} \binom{k-m}{m} p^m q^{k-2m} P_{n,k-m}, \tag{4}$$

where $[k/2]$ stands for the integer part of $k/2$ and $\binom{n}{m}$ is the binomial coefficient. The latter equation can be written in a matrix form as a transfer matrix relation:

$$|\mathbf{P}_{n+1}\rangle = \mathbf{T}|\mathbf{P}_n\rangle, \tag{5}$$

where

$$T_{j,k} = \binom{k}{j-k} p^{j-k} q^{2k-j} \quad \text{and} \quad \langle \mathbf{P}_n| = (P_{n,1}, P_{n,2}, \ldots)\,.$$

The solution of Eq. (5) is simply

$$|\mathbf{P}_n\rangle = \mathbf{T}^n|\mathbf{P}_0\rangle\,, \tag{6}$$

where $|\mathbf{P}_0\rangle$ is an initial vector. Since Eq. (2) represents the same problem it means that Eq. (6) is its general solution for the case $p+q=1$. However, the expression (4) is defined for *any* p and q (not only for $p+q=1$) thus the analytic continuation of (6) is a solution of Eq. (2) for arbitrary p and q. It is interesting to note that the first approach leads to a non-linear relation (Eq. (2)) while the second approach to the same problem leads to a solvable linear recursion (Eq. (4)).

Now we observe that the solution of the auxiliary problem can be related to the logistic map (1). Let us suppose that the function f_n has a "hidden" continuous dependence, say, $f_n \equiv f_n(x) = \sum_k F_{n,k} x^k$. Then, this function, $F_{n,k}$, obeys a convolution relation

$$F_{n+1,k} = \lambda F_{n,k} - \lambda \sum_{k'} F_{n,k'} F_{n,k-k'} , \tag{7}$$

which is a special case of Eq. (2) with $q = -p = \lambda$. Thus, the general solution of Eq. (7) is (6) with substitutions $q \to \lambda$ and $p \to -\lambda$. Namely,

$$|\mathbf{F}_n\rangle = \mathbf{T}^n |\mathbf{F}_0\rangle, \qquad \text{with } T_{j,k} = (-1)^{j-k} \binom{k}{j-k} \lambda^k, \tag{8}$$

where $\langle \mathbf{F}_n| = (F_{n,1}, F_{n,2}, \dots)$.

To complete the derivation one should perform an inverse transform, i.e., to find the "generating" function f_n. This is done by multiplying a ket-vector $\langle \mathbf{F}_n|$ by a bra-vector of powers of x, $\langle \mathbf{x}| = (x, x^2, x^3, \dots)$. Thus,

$$f_n = \langle \mathbf{x}|\mathbf{T}^n|\mathbf{F}_0\rangle.$$

To get rid of arbitrary coefficients $F_{0,k}$, one should recall that $f \equiv f_0 = \sum_k F_{0,k} x^k$. The simplest (but not the unique!) choice is $\langle \mathbf{F}_0| = \langle \mathbf{e}_1| = (1, 0, 0, \dots)$. Then, $f = x$ and the solution of Eq. (1) takes the form

$$f_n = \langle \mathbf{f}|\mathbf{T}^n|\mathbf{e}_1\rangle , \tag{9}$$

where $\mathbf{T}$ is defined by Eq. (8) and $\langle \mathbf{f}| = (f, f^2, f^3, \dots)$. Correspondingly, a *fix point equation* for the mapping (9) reads

$$f = \langle \mathbf{f}|\mathbf{T}^n|\mathbf{e}_1\rangle .$$

The conditions for the existence of the solution of this equation give the well-known bifurcation points of the period doubling.

The transfer matrix $\mathbf{T}$ can be naturally decomposed into combinatoric and λ-component matrices: $\mathbf{T} = \mathbf{C}\Lambda$, where $C_{j,k} = (-1)^{j-k}\binom{k}{j-k}$ and $\Lambda_{j,k} = \delta_{j,k}\lambda^k$. These matrices are infinite but the calculation of their nth power requires only finite $2^n \times 2^n$ upper-left corners of them. Since $\mathbf{T}$ is a triangular matrix it follows that all its eigenvalues are: λ^k $(k = 1, \dots, 2^n)$.

The ultimate aim of these manipulations is to study the λ-dependence of the mapping (1) in the limit $n \to \infty$. A possible approach to it may be as follow. Let us construct a generating function $\phi(y) = \sum_n^\infty f_n y^n$. Then,

$$\phi(y) = \langle \mathbf{f}|(\mathbf{I} - y\mathbf{T})^{-1}|\mathbf{e}_1\rangle , \tag{10}$$

where $\mathbf{I}$ is the unit matrix. This calculation is closely related to an eigenvectors problem. If it were possible to calculate the resolvent $(\mathbf{I} - y\mathbf{T})^{-1}$ one could proceed with a steepest descent of Cauchy integral

$$f_n = \frac{1}{2\pi i} \oint_{C_0} \frac{\phi(y)}{y^{n+1}} dy , \tag{11}$$

where the contour C_0 includes the origin of coordinates.

However, this approach may face a serious problem. The resolvent is an infinite matrix with diagonal elements (eigenvalues) equal to $(1 - y\lambda^k)^{-1}$ $k = 1, 2, \ldots$, i.e., they have poles at $y_k = \lambda^{-k}$. For $\lambda > 1$ this infinite set has a limit point, $y = 0$. If the first column of the resolvent has the same property, the function $\phi(y)$ may have an essential singularity at $y = 0$. Then, (10) does not exist there and Eq. (11) becomes unapplicable. This fine point should be studied carefully.

One more observation can be done: this approach is trivially generalizable for the case of an n-dependent function λ, i.e. for the equation $f_{n+1} = \lambda_n f_n(1 - f_n)$. Then, the analog of (9) will be written as

$$f_n = \langle \mathbf{f} | \mathbf{T}_1 \mathbf{T}_2 \cdots \mathbf{T}_n | \mathbf{e}_1 \rangle ,$$

where $\mathbf{T}_\ell = \mathbf{C}\Lambda_\ell$ and $(\Lambda_\ell)_{j,k} = \delta_{j,k}(\lambda_\ell)^k$.

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