Fractal measures of diffusion in the presence of random fields

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We study diffusion in a linear chain where random fields are located at each site and can accept the values $\pm E$ with equal probability. While the average density distribution of a random walker scales and is described by a single exponent, it requires an infinite hierarchy of exponents α to characterize the fluctuations. Their density distribution $f(\alpha)$ is a single-hump function and depends continuously on the magnitude of the field E.

The problem of anomalous diffusion in disordered systems has been studied extensively in recent years.[1-8] Typically, the mean-square displacement $\langle R^2(t)\rangle$ of a random walker follows a power-law behavior $\langle R^2(t)\rangle \sim t^{2/d_w}$. In uniform systems, $d_w=2$ (Fick's law), while in self-similar fractals the motion of the walker is slowed down in all length scales and $d_w>2$.

"Ultra"-anomalously slow motion characterized by a *logarithmic* time dependence for $\langle R^2(t)\rangle$ was found by Sinai[1] when considering diffusion in a linear chain in the presence of random fields. In order to obtain a deeper understanding of the underlying dynamics and to characterize the process further we have studied the fractal measures[9-12] of the random walk. We find that while the average density distribution $\langle P(x,t)\rangle$ scales with a single exponent, it requires an infinite hierarchy of exponents to describe the fluctuations of $P(x,t)$. These exponents α and in particular their density distribution $f(\alpha)$ represent the essential quantities to characterize the dynamical process. We have found that $f(\alpha)$ is a single-hump function and is similar in shape to those found to characterize a large variety of other "multifractal" phenomena,[9-15] including fluid turbulence,[9,10] kinetic aggregation,[12-14] and voltage drops in percolation systems.[15]

Consider a linear chain with random fields applying at each site. The transition rates $w_{i,i\pm1}$ between neighboring sites are

$$W_{i,i\pm1}=\tfrac{1}{2}(1\pm E_i)\ , \tag{1}$$

where the random fields E_i can accept the values $\pm E$ with equal probability. For this system Sinai[1] found the remarkable result that the mean-square displacement $\langle X^2(t)\rangle$ of a random walker follows asymptotically

$$\langle X^2(t)\rangle \sim \ln^4 t\ . \tag{2}$$

The origin of this logarithmic slow motion lies in the fact that the walker can get easily stuck in compensated regions of the chain where fields of opposite directions point to the same site ("hot" sites, see Fig. 1).

An important quantity to characterize the walk is the density distribution $P(x,t)$, defined as the probability to find the walker at time t in distance x from its starting point. First we study the mean density distributions $\langle P(x,t)\rangle$ averaged over the ensemble of random chains with transition rates (1). We have used the exact enumeration method[6,16] which enables us to enumerate exactly $P(x,t)$ for single configurations. Since the characteristic length scales as $\ln^2 t$, Eq. (2), the simplest scaling assumption for $\langle P(x,t)\rangle$ is (see also Ref. 17)

$$\langle P(x,t)\rangle = \ln^{-2} t\, f(x/\ln^2 t)\ . \tag{3}$$

The prefactor $\ln^{-2}t$ is due to normalization. In Fig. 2 we have plotted $\langle \mathbf{P}(x,t)\rangle \ln^2 t$ as a function of $x/\ln^2 t$ for various values of x and t. The data collapse strongly supports the scaling ansatz. This indicates that essentially one exponent is needed to characterize the *mean* density distribution. The corresponding moments $\langle X^n\rangle$ scale as $\langle (X^2)^{n/2}\rangle \sim \ln^{2n} t$.

Next we consider the fluctuations in the single-chain probabilities $P(x,t)$ by studying their fractal measures. We define a partition function

$$Z(q,t)=\sum_{i=-x(t)}^{+x(t)} P(i,t)^q \sim x(t)^{-\tau(q)}\ , \tag{4}$$

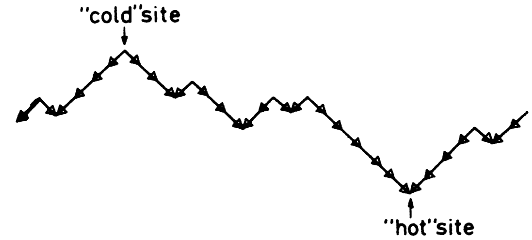

FIG. 1. Sketch of the linear chain with random fields $E_i=\pm E$ at each site i. Fields pointing to the left and right are denoted by arrows ↙ and ↘ , respectively. The valleys represent hot sites where the walker can get stuck, while the peaks cold sites where the walker is not likely to stay.

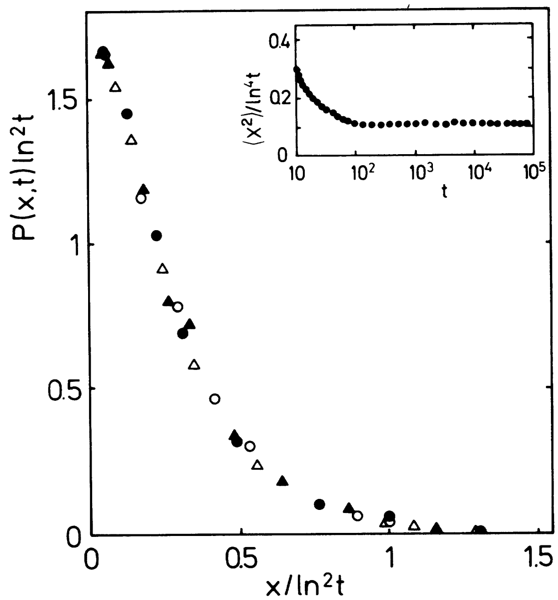

FIG. 2. $\langle p(x,t)\rangle \ln^2 t$ as a function of $x/\ln^2 t$ for $E=0.7$ at times $t=10^4$(○), 2×10^4 (△), 4×10^4 (●), and 10^5 (▲) for values of x between 5 and 150. To obtain $\langle P(x,t)\rangle$, we used the exact enumeration method and averaged over 1000 random chain configurations, each consisting of 1000 sites. To show the convergence to the asymptotic time regime, we have plotted also $\langle X^2\rangle/\ln^4 t$ in the inset. After $\sim 10^2$ time steps, the curve approaches a constant, in excellent agreement with Sinai's prediction (2).

where $x(t)$ is the mean displacement on the considered chain. Equation (4) defines the set of exponents $\tau(q)$. A similar ansatz has been used, e.g., in studies of chaotic dynamical systems[11] or in kinetic aggregation.[13]

Following Ref. 13 we rewrite (4) as

$$Z(q)=\int p^q n(p) d\ln p \ , \tag{5}$$

where $n(p)d\ln p$ is the number of sites for which $\ln p$ is in the range ($\ln p$, $\ln p + d\ln p$). For general q, the integration in (5) is dominated by some value $p=p^*(q)$ which maximizes the integrand $p^q n(p)$ and depends explicitly on q. Correspondingly, for a given q value a certain subset $i(q)$ dominates the sum (3) and $p^*(q)$ is the characteristic value of $p(i,t)$ for $i\in i(q)$.

Following Halsey *et al.*[11] we write the scaling ansatz $p^* \sim X^{-\alpha(q)}$ and $n(p^*)\sim X^{f(q)}$ which define the scaling exponents $\alpha(q)$ and $f(q)$. By definition, $\alpha(q)$ is the singularity strength of the probabilities and $f(q)$ denotes their density distribution. From (4) we obtain

$$Z(q)=n(p^*)p^* \sim x^{-[q\alpha(q)-f(q)]} \tag{6}$$

and hence

$$\tau(q)=q\alpha(q)-f(q) \ . \tag{7}$$

In the limit $E=0$ one has

$$P(x,t)\sim t^{-1/2}\exp(-x^2/t)$$

and (4) becomes

$$Z(q)=\int_{-x}^{x} p^q(x',t)dx' \sim x^{-(q-1)} \ . \tag{8}$$

This gives $\tau(q)=q-1$ and hence $\alpha=f(\alpha)=1$. In the limit $E\rightarrow 1$ the hot sites represent traps from which the walker cannot escape. Hence, after a few time steps we have

$$P(x,t)=\delta_{x,x_0} \ , \tag{9}$$

where x_0 is the coordinate of the trap closest to the origin of the walk. Substituting (9) in (4) yields $\tau(q)=0$ for $q\geq 0$ and $\tau(q)=-\infty$ for $q<0$. The two limiting cases are shown in Fig. 3.

For the case of general E we have calculated $P(x,t)$ for each chain by the exact enumeration method. From (4) we obtain $\tau(q)$ for a single chain, which then was averaged over many random configurations. We found that for large times t the exponents $\tau(q)$ were independent of t. The results for $\tau(q)$ for several values of the fields are shown in Fig. 3(a). The figure suggests that the whole hierarchy of exponents changes *continuously* with E be-

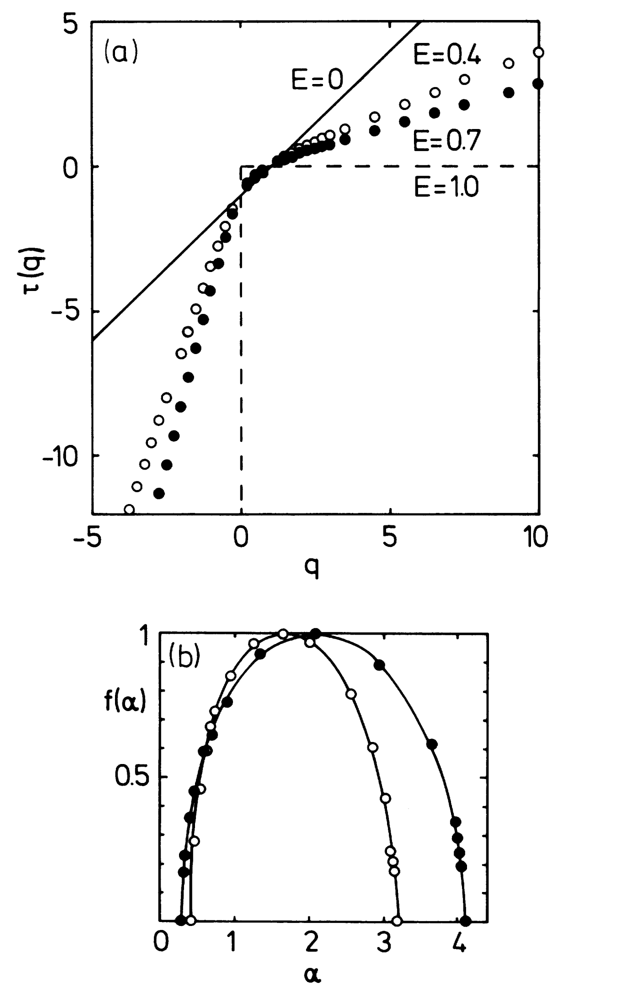

FIG. 3. (a) The exponents $\tau(q)$ vs q for several values of E. The results represent averages over 200 lattice configurations each. (b) The density distribution of the singularity strength $f(\alpha)$ vs α for $E=0.4$(○) and 0.7(●). $\alpha_{\min}$ and $\alpha_{\max}$ represent the asymptotic slopes ($q\rightarrow\pm\infty$) of $\tau(q)$ [(a)].

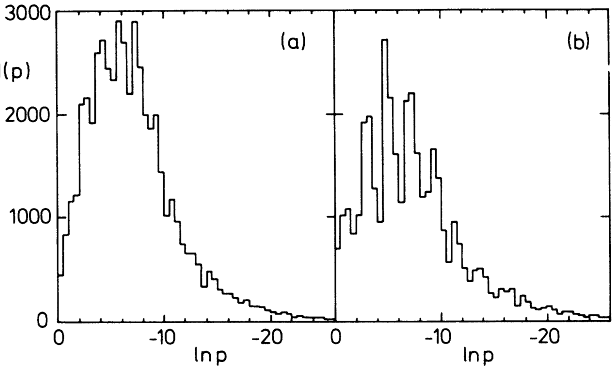

FIG. 4. The histogram $n(p)$ of $P(x,t)$ for (a) $E=0.7$ and (b) $E=0.8$ at $t=10^4$ for 2000 configurations each. Note that $n(p)$ is significantly broader than the corresponding $n(p)$ for DLA (Ref. 13) and percolation (Ref. 15).

tween the two limiting cases $E=0$ and $E=1$. This is to be compared with the exponent of $\langle X^2 \rangle$ in Sinai's law Eq. (2) which is independent of the field E.

It is conventional to also calculate the Legendre transform with respect to q of $\tau(q)$, $f(\alpha)=q\alpha-\tau(q)$, where $\alpha=d\tau/dq$. The minimum and maximum values $\alpha_{\min}$ and $\alpha_{\max}$ of the singularity strength α are obtained from the asymptotic slopes of $\tau(q)$ for $q\rightarrow+\infty$ and $q\rightarrow-\infty$, respectively; $\alpha_{\min}$ describes how the probabilities of the hottest sites of the chain scale with x, while from $\alpha_{\max}$ we obtain information on those regimes which have very small probability to be visited ("cold" sites). For $E=0$, we have $f(\alpha)=\delta_{\alpha,1}$ and hence $\alpha_{\min}=\alpha_{\max}=1$, indicating the absence of hot and cold sites. In the case of $E\rightarrow 1$ we find $\alpha_{\min}=0$, $\alpha_{\max}=\infty$, representing the maximum separation between hot and cold sites. In Fig. 3(b) we show $f(\alpha)$ versus α for several values of E. The functions $f(\alpha)$ have a single-hump shape. With increasing field E, $\alpha_{\max}$ increases and $\alpha_{\min}$ decreases.

From $P(x,t)$ with x between $-x(t)$ and $+x(t)$ [see Eq. (4)] we determine the distribution function $n(p)$. A representative result is shown in Fig. 4. For $E=0.7$, the function has a pronounced and broad-structured maximum which for $E=0.8$, splits into several well-separated peaks. In both cases, there exists a long tail which extends to the extremely small values of $P(x,t)$ at the cold sites of the chain. The form of $n(p)$ for $E=0.7$ resembles (although it is much broader) the shape of the growth probability distribution in kinetic aggregation[13,14] and the voltage distribution in random resistor networks.[15] In fact, there is a close analogy between these systems and the problem of diffusion in random fields. The function $P(x,t)$ considered here is analogous to the growth site probability distribution p_i in diffusion-limited aggregation (DLA), which is defined as the probability that a given site i of the perimeter sites becomes part of the aggregate in the next time step. In DLA the sites at the tips of the cluster can be occupied more easily, and p_i is large (hot tips). In contrast, very few particles can get deep inside the fjords, where p_i is extremely small. A similar role is played by the voltage drops across the bonds in random resistor networks.

In summary, we have studied the fluctuations of the density distribution in the presence of random fields within a range $x(t)$ around the origin of the random walker. We have found that they are described by an infinite hierarchy of exponents rather than by a single gap exponent. We also checked how the cutoff $x(t)$ and the type of averaging used here affect the main result for $f(\alpha)$. To this end, we varied the cutoff between $x(t)/2$ and $2x(t)$ and found that the shape of $f(\alpha)$ remained essentially unchanged. We also calculated $\tau(q)$ using Eq. (4) by averaging $Z(q)$ over many configurations. In this case we found slight deviations from Fig. 3, which leaves the question if we reached the asymptotic regime or not, open. The question of whether the multifractality is a feature of the dynamical process or/and of the steady state (as for the voltage drop distribution in percolation) is also open. We know,[17] however, that in the case of diffusion on percolation networks (without bias) both the dynamical process and the steady state might be of different nature. While the steady state is equivalent to the voltage drop problem and thus should also show multifractality, the dynamical process is described by $Z(q,t)\sim x^{-d_f(q-1)}$ and thus is governed by a single gap exponent.[18]

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[1]Ya. Sinai, in *Proceedings of the Berlin Conference on Mathematical Problems in Theoretical Physics,* edited by R. Schrader, R. Seiler, and D. A. Uhlenbrock (Springer, Berlin, 1982), p. 12.
[2]B. Derrida and Y. Pomeau, Phys. Rev. Lett. **48**, 627 (1982).
[3]S. Alexander and R. Orbach, J. Phys. (Paris) **43**, L625 (1982).
[4]D. Stauffer, *Introduction to Percolation Theory* (Taylor and Francis, London, 1985).
[5]Y. Gefen, A. Aharony, and S. Alexander, Phys. Rev. Lett. **50**, 77 (1983).
[6]For a recent review see S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1987).
[7]A. Bunde, S. Havlin, H. E. Stanley, B. Trus, and G. H. Weiss, Phys. Rev. B **34**, 8129 (1986).
[8]S. Havlin, A. Bunde, Y. Glaser, and H. E. Stanley, Phys. Rev. A **34**, 3492 (1986).
[9]B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).
[10]H. G. E. Hentschel and I. Procaccia, Physica **8D**, 435 (1983).
[11]T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and

B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).
[12]T. C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**, 854 (1986).
[13]C. Amitrano, A. Coniglio, and F. di Liberto, Phys. Rev. Lett. **57**, 1016 (1986).
[14]J. Nittmann, H. E. Stanley, E. Touboul, and G. Daccord, Phys. Rev. Lett. **58**, 619 (1987).
[15]L. de Arcangelis, S. Redner, and A. Coniglio, Phys. Rev. B **31**, 4725 (1985); **34**, 4656 (1986).
[16]S. Havlin, A. Bunde, and J. Kiefer, J. Phys. A **19**, L419 (1986).
[17]E. Koscielny-Bunde, A. Bunde, S. Havlin, and H. E. Stanley, Phys. Rev. A **37**, 1821 (1988).
[18]A. Bunde, H. Harder, S. Havlin, and H. E. Roman, J. Phys. A **20**, L865 (1987).