The theory of the firm is of considerable interest in economics [1–18]. In its standard form, it focuses primarily on the technology of production. The traditional dynamic mechanisms built into the theory of the firm are investment in physical capital and research & development (see e.g. [18] and references therein). In recent years, richer models have been formulated in which learning and the development of organizational infrastructure are equally important sources of firm dynamics [10–20]. Using the Compustat database over the time period 1975-1991, we find: (i) the distribution of the logarithm of the annual growth rates for firms with approximately the same sales displays an exponential form, and (ii) the fluctuations in the growth rates—measured by the width of this distribution—scale as a power law with firm sales. We place these findings of scaling behavior in the context of conventional economics by considering firm growth dynamics with temporal correlations and also by considering a hierarchical organization of the departments of a firm.

The simplest model for firm growth was proposed by Gibrat [1]. Its basic assumptions are that the rate of firm growth is (i) independent of firm size (law of proportionate effect), and (ii) uncorrelated in time. These assumptions can be formalized by the following random
multiplicative process: $S_{t+\Delta t} = S_t(1 + \epsilon_t)$, where $S_{t+\Delta t}$ and $S_t$ are the sales of the firm at time $t + \Delta t$ and $t$ respectively, and $\epsilon_t$ is an uncorrelated random number with mean close to zero and standard deviation much smaller than one. Hence $\log S_t$ follows a simple random walk so that firm sizes are log-normally distributed. Also, for sufficiently large time intervals $T \gg \Delta t$, the growth rates $S_{t+T}/S_t$ are log-normally distributed.

While it is known that Gibrat’s assumptions are rejected empirically, many theoretical and empirical analyzes still use the Gibrat model as a benchmark, for lack of a better alternative. To achieve a more realistic characterization of the firm dynamics, we analyze the statistical properties of the growth rates.

We studied all US manufacturing publicly-traded firms within the years 1975-1991. The data were taken from the Compustat database and all values for sales have been adjusted to 1987 dollars by the GNP price deflator. We define a firm’s annual growth rate as $R \equiv S_1/S_0$, where $S_0$ and $S_1$ are its sales in two consecutive years.

It is customary to study firm growth on logarithmic scales, so we define $r \equiv \ln(S_1/S_0)$ and $s_0 \equiv \ln S_0$ and calculate the conditional distribution $p(r|s_0)$ of growth rates $r$ with a given initial sales value $s_0$.

The distribution $p(r|s_0)$ of the growth rates from 1990 to 1991 is shown in Fig. 1a for two different values of initial sales. Remarkably, both curves display a simple “tent-shaped” form. The distribution is not Gaussian — as expected from the Gibrat model — but rather is exponential,

$$p(r|s_0) = \frac{1}{\sqrt{2\pi}\sigma(s_0)} \exp\left(-\frac{\sqrt{2}|r - \bar{r}(s_0)|}{\sigma(s_0)}\right).$$

The straight lines shown in Fig. 1a are calculated from the average growth rate $\bar{r}(s_0)$ and the standard deviation $\sigma(s_0)$ obtained by fitting the data set to Eq. (1).

We also find that the data for each annual interval from 1975-91 fit well to Eq. (1), with only small variations in the parameters $\bar{r}(s_0)$ and $\sigma(s_0)$. To improve the statistics, we therefore calculate the new distribution by averaging all the data from the 16 annual intervals in the database. As shown in Fig. 1b, the data now scatter much less and the
shape is well described by Eq. (1). For this reason, we have also included in the figure data for “volatile” cases, corresponding to sales of only about $2.6 \times 10^5$ dollars.

As is apparent from Fig. 1b, $\sigma(s_0)$ decreases with increasing $s_0$. We find $\sigma(s_0)$ is well approximated over more than 7 orders of magnitude—from sales of less than $10^4$ dollars up to sales of more than $10^{11}$ dollars—by the law

$$\sigma(s_0) = a \exp(-\beta s_0) = a s_0^{-\beta}, \quad (2)$$

where $a \approx 6.66$ and $\beta = 0.15 \pm 0.03$ (Fig. 2).

We performed a parallel analysis for the number of employees, and the corresponding standard deviation is shown in Fig. 2. The data are linear over roughly 5 orders of magnitude, from firms with only 10 employees to firms with almost $10^6$ employees. The slope $\beta = 0.16 \pm 0.03$ is the same, within error bars, as that found for sales.

We find that Eqs. (1) and (2) accurately describe three additional indicators of firm growth, (i) cost of goods sold (with exponent $\beta = 0.16 \pm 0.03$) (ii) assets ($\beta = 0.17 \pm 0.04$) and (iii) property, plant & equipment ($\beta = 0.18 \pm 0.03$).

What is remarkable about Eqs. (1) and (2) is that they govern the growth rates of a diverse set of firms. They range not only in their size but also in what they manufacture. The conventional economic theory of the firm is based on production technology, which varies from product to product. Conventional theory does not suggest that the processes governing the growth rate of car companies should be the same as those governing, e.g., pharmaceutical or paper firms. Indeed, our findings are reminiscent of the concept of universality found in statistical physics, where different systems can be characterized by the same fundamental laws, independent of “microscopic” details.

In statistical physics, scaling phenomena of the sort that we have uncovered in the sales and employee distribution functions are sometimes represented graphically by plotting a suitably “scaled” dependent variable as a function of a suitably “scaled” independent variable. If scaling holds, then the data for a wide range of parameter values are said to “collapse” upon a single curve. To test the present data for such data collapse, we plot.
(Fig. 3) the scaled probability density $p_{\text{scal}} \equiv \sqrt{2} \sigma(s_0)p(r|s_0)$ as a function of the scaled growth rates of both sales and employees $r_{\text{scal}} \equiv \sqrt{2}[\bar{r} - \bar{r}(s_0)]/\sigma(s_0)$. The data collapse upon the single curve $p_{\text{scal}} = \exp(-|r_{\text{scal}}|)$. Our results for (i) cost of goods sold, (ii) assets, and (iii) property, plant & equipment are equally consistent with such scaling.

The Gibrat model, which yields a log-normal distribution of the growth rates for sufficiently long time intervals, fails to explain the observed distribution of annual growth rates (even for intervals as long as 5 years, we find $p(r|s_0)$ does not obey a normal distribution.) There is, however, a simple dynamic process in which successive values of $S_t$ are correlated that generates the observed tent-shaped distribution. Suppose each firm has a tendency to maintain a value $S^*$, which evolves only slowly in time and which can be interpreted as the minimum point of a “U-shaped” average cost curve in conventional economic theory. This type of dynamics is similar to what is known in economics as regression towards the mean [25,26]. If the growth process has a constant “back-drift,” i.e.

$$S_{t+\Delta t}/S_t = \begin{cases} 
  k(1 + \epsilon_t), & \text{for } S_t < S^*, \\
  \frac{1}{k}(1 + \epsilon_t), & \text{for } S_t > S^*,
\end{cases}$$

(3)

where $k$ is a constant larger than one, then the distribution of growth rates is the tent-shaped distribution Eq. (1) with a width proportional to $1/\ln k$ [28].

Our empirical findings of Eq. (2) are consistent with a hierarchical model of the internal structure of each firm. In zeroth order approximation, suppose that a given firm consists of independent units. If the unit’s sales fluctuate with a standard deviation independent of $s_0$, then Eq. (2) follows with $\beta = 1/2$. The much smaller empirical value of $\beta$ that we find indicates the presence of strong, positive correlations among the firm’s units. We propose a model relying on a technology of management (which may be common across firms) as opposed to a technology of production; this model may lead to some insight into why the behavior of apparently diverse firms follows a simple law.

Consider a tree-like hierarchical organization of a firm [17]. The root of the tree represents the head of the firm, whose policy is processed to the level beneath, and so on, until finally the $N = z^n$ lowest-level units take action; here $z$ is the average number of links connecting
the levels and \( n \) the average number of levels. The \( N \) lowest-level units have sales \( \xi_i \) and mean \( \langle \xi \rangle \), so \( S_0 = \sum_{i=1}^{N} \xi_i = N \langle \xi \rangle \).

Suppose that the head of the firm suggests a policy with the intention to change the sales of each lowest-level unit by an amount \( \Delta \xi \). If this policy were to be propagated through the hierarchy without any modifications, then the change in sales would be \( \Delta S = N \Delta \xi = S_0 \Delta \xi / \langle \xi \rangle \). Accordingly, \( r = \ln[(S_0 + \Delta S)/S_0] = \ln[1 + \Delta \xi / \langle \xi \rangle] \), which is independent of \( S_0 \). It follows that \( \beta = 0 \).

More realistically, each unit is not only influenced by the policy of the head but also by other (external and internal) factors. An example is that different levels have different types of information. Managers at each level might deviate from decisions made higher up in the tree if other information suggests to them that another action is appropriate. Another reason for a modification of the policy is organizational failure, due either to poor communication or disobedience. For these reasons, we assume that each manager follows his supervisor’s policy with a probability \( \Pi \), while with probability \( (1 - \Pi) \) imposes a new independent policy for his subunits. Straightforward calculation using methods described, e.g., in [29] yields Eq. (2) for \( n \gg 1 \), with exponent \( \beta \) given by the simple formula

\[
\beta = \begin{cases} 
- \ln \Pi / \ln z & \text{if } \Pi > z^{-1/2}, \\
1/2 & \text{if } \Pi < z^{-1/2}
\end{cases}
\]  

(4)

(for small \( n \), Eq. (4) is still a good approximation—e.g., for \( n = 3 \) and \( z = 2 \), the deviation from the value \( \beta = 0.20 \) is only 0.03). Equation (4) is confirmed in the two limiting cases: when \( \Pi = 1 \) (absolute control) \( \beta = 0 \), while for all \( \Pi < 1/z^{1/2} \), decisions at the upper levels of management have no statistical effect on decisions made at lower levels, and \( \beta = 1/2 \). Moreover, for a given value of \( \beta < 1/2 \) the control level \( \Pi \) will be a decreasing function of \( z \): \( \Pi = z^{-\beta} \). For example, if we choose the empirical value \( \beta \approx 0.15 \), then Eq. (4) predicts the plausible result \( 0.9 \geq \Pi \geq 0.7 \) for a range of \( z \) in the interval \( 2 \leq z \leq 10 \).

Our central results, Eqs. (1) and (2), constitute a test that any accurate theory of the firm must pass, and support the possibility [30] that the scaling laws used to describe complex but inanimate systems comprised of many interacting particles (as occurs in many physical
systems) may be usefully extended to describe complex but animate systems comprised of many interacting subsystems (as occurs in economics).
REFERENCES


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FIG. 1. (a) Probability density \( p(r|s_0) \) of the growth rate \( r \equiv \ln(S_1/S_0) \) from year 1990 to 1991 for all publicly-traded US manufacturing firms in 1994 Compustat with standard industrial classification index of 2000-3999. We examine 1991 because between 1992 and 1994 there are several firms with zero sales that either have gone out of business or are "new technology" firms (developing new products). We show the data for two different bins of initial sales (with sizes increasing by powers of 4): \( 4^{11.5} < S_0 < 4^{12.5} \) and \( 4^{14.5} < S_0 < 4^{15.5} \). Within each sales bin, each firm has a different value of \( R \), so the abscissa value is obtained by binning these \( R \) values. The solid lines are fits to Eq. (1) using the mean \( \bar{r}(s_0) \) and standard deviation \( \sigma(s_0) \) calculated from the data. (b) Probability density \( p(r|s_0) \) of the annual growth rate, for three different bins of initial sales: \( 4^{8.5} < S_0 < 4^{9.5} \), \( 4^{11.5} < S_0 < 4^{12.5} \), and \( 4^{14.5} < S_0 < 4^{15.5} \). The data were averaged over all 16 one-year periods between 1975-1991. The solid lines are fits to Eq. (1) using the mean \( \bar{r}(s_0) \) and standard deviation \( \sigma(s_0) \) calculated from all data.

FIG. 2. Standard deviation of the one-year growth rates of the sales and of the one-year growth rates of the number of employees as a function of the initial values. The solid lines are least square fits to the data with slopes \( \beta = 0.15 \pm 0.03 \) for the sales and \( \beta = 0.16 \pm 0.03 \) for the number of employees. We also show error bars of one standard deviations about each data point. These error bars appear asymmetric since the ordinate is a log scale.

FIG. 3. Scaled probability density \( p_{scal} \equiv 2^{1/2} \sigma(s_0) p(r|s_0) \) as a function of the scaled growth rate \( r_{scal} \equiv 2^{1/2}[r - \bar{r}(s_0)]/\sigma(s_0) \). The values were rescaled using the measured values of \( \bar{r}(s_0) \) and \( \sigma(s_0) \). Also we show the analogous scaled quantities for the number of employees. All the data collapse upon the universal curve \( p_{scal} = \exp(-|r_{scal}|) \) as predicted by Eqs. (1) and (2).