

Approximate theory of photon migration in a two-layer medium

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We recently examined properties of the surface emission profiles of particles (photons) injected into a turbid medium consisting of two layers [R. Nossal, J. Kiefer, G. H. Weiss, R. F. Bonner, H. Taitelbaum, and S. Havlin, "Photon Migration in Layered Media," *Appl. Opt.* **27**, 3382-3391 (1988)]. The two layers differ in the coefficient that appears when internal absorption is modeled in terms of Beer's law. The model relates to the injection of laser radiation into tissue for diagnostic or therapeutic purposes. Results of our earlier work were derived from extensive computer simulations. In the present paper we discuss a simple analytical approximation to the surface intensity profile valid when the absorptivity of the upper layer is greater than that of the lower.

I. Introduction

In two recent articles^{1,2} we described a model of photon migration in a turbid media (hereafter, because of its applications, referred to as a tissue) based on a lattice random walk model with two adjustable parameters, L , the lattice spacing, and μ , an absorption coefficient. In the first of these articles we assumed that the tissue was infinite in extent, the scattering isotropic, and just the pair of parameters, L and μ , are needed to characterize the entire tissue, i.e., any inhomogeneities are strictly local ones that give rise to scattering. Results of the theory were shown to be in good agreement with data obtained using laser radiation into human tissue. The second reference treats the case of a two-layer model in which the absorption coefficients in the two layers differ, but the lattice spacings in both layers are the same. This model may be useful in analyzing, for example, the penetration of pigmented skin by laser radiation or the penetration of tissue covered by a melanoma. While mathematical results for useful physical parameters have been found in convenient analytical form in the case of a single layer medium, we presented no mathematical expression for the intensity of reflected radiation in the two-layer medium, relying instead on simulation results based on the exact enumeration³ method. In the present paper we present an approximate analysis of

the two-layer medium leading to a readily calculable intensity profile. This will be shown to be in good agreement with our earlier simulated data when absorption in the surface layer is greater than that in the lower layer. In the contrary case, it has been shown that there are no features that appear in the profile of reflected intensity that would allow one to distinguish between a homogeneous tissue and a two-layer model on the basis of measurements of surface intensities alone.²

The particular model to be analyzed is shown in Fig. 1 in which photons are injected into a tissue consisting of two layers at the point marked (0,0,0). The coordinates (x,y,z) are chosen so that (x,y) are measured transverse to the injected beam, while z is measured in the direction of the beam. The positive z -direction points into the tissue, and the (uniform) thickness of the upper layer is taken equal to D . The phenomenological absorption coefficients μ_i ($i = 1,2$) are defined in terms of the probabilities that a photon will survive a single step on the lattice without being absorbed. These probabilities will be expressed as $\exp(-\mu_i)$, where the subscript i is equal to 1 or 2 according to whether the photon is found in the upper or lower layer. After injection, the photons diffuse randomly within the tissue. The mechanism for producing diffusive motion is that of scattering by either inhomogeneities within the tissue or red blood cells. Photons, once injected, eventually either reach the upper surface where the intensity of the reflected radiation can be measured, or they suffer internal absorption. Motion within the tissue is modeled by a nearest-neighbor random walk on a simple cubic lattice. The upper surface of the tissue is assumed to be totally absorbing, and it will be assumed that any photon reaching the

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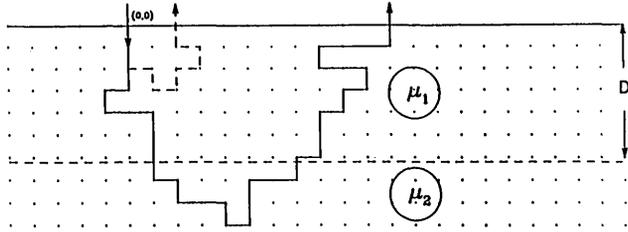


Fig. 1. Two-dimensional projection of a semi-infinite medium consisting of two layers. The thickness of the upper layer is taken equal to D , and the absorption coefficient in the upper and lower layers, respectively, are taken equal to μ_1 and μ_2 .

surface at $z = 0$ contributes to the measured intensity, the photon thereafter disappearing from the system. Results found using the lattice random walk model for homogeneous tissue have been shown to agree with a more realistic continuum model of the diffusion process, thereby lending some credibility to the use of a seemingly nonphysical model. This suggests the desirability of being able to express the theoretical results in computable form rather than relying on essentially qualitative information furnished by simulations. The function of immediate interest in the present context is the distribution of reflected intensity on the tissue surface as a function of distance from the point of injection, since this constitutes the measurable information. A second function of direct physical interest¹ is $\langle n|\rho \rangle$, which is the expected path length traveled by a photon that ultimately emerges at the surface at a distance ρ from the injection point.

II. Two-Layer Medium

One naturally expects that intensity measurements containing information about the nature of the tissue can only be regarded as being able to furnish reliable parameters when the two media are quite dissimilar because of measurement noise. There are two cases to be considered according to whether the upper layer has the greater absorptivity or the lower, since the qualitative nature of the photon trajectory depends strongly on the particular arrangement of absorptivities.²

A. Pigmented Upper Layer ($\mu_1 > \mu_2$)

As in the uniform case it is convenient to analyze photon migration in terms of a random walk constrained to move on a lattice. Consider a photon reaching the surface, $z = 0$, at the n th step of the random walk. The total number of steps can be decomposed into contributions from steps in the upper and lower layers, n_1 and n_2 , respectively, so that

$$n_1 + n_2 = n. \quad (1)$$

The joint probability that a photon will reach the surface at step n , reaching it at a distance $\rho (= \sqrt{x^2 + y^2})$ from the point of injection, will be denoted by $\Gamma(n_1, n_2, \rho | \mu_1, \mu_2)$ in analogy to the function $\Gamma(n, \rho | \mu)$ calculated for the uniformly homogeneous medium in Ref. 1. In that case one can decompose $\Gamma(n, \rho | \mu)$ into a product of two terms, one of which accounts for diffusion in the absence of absorption and the second of

which accounts only for the internal absorption. This decomposition takes the form

$$\Gamma(n, \rho | \mu) = F(n, \rho) \exp(-\mu n), \quad (2)$$

where the expression for $F(n, \rho)$, calculated for the nearest-neighbor random walk, has been shown¹ to be approximately equal to

$$F(n, \rho) \approx \frac{\sqrt{3}}{2(2\pi n)^{3/2}} [1 - \exp(-6/n)] \exp\left(-\frac{3\rho^2}{2n}\right). \quad (3)$$

In the two-layer model it is also natural to assume that $\Gamma(n_1, n_2, \rho | \mu_1, \mu_2)$ can be decomposed in the same way, allowing us to write

$$\begin{aligned} \Gamma(n_1, n_2, \rho | \mu_1, \mu_2) &= F(n, \rho) \exp[-\mu_1 n_1 - \mu_2 n_2] \\ &= F(n, \rho) \exp[-\mu_2 n - n_1(\mu_1 - \mu_2)], \end{aligned} \quad (4)$$

since the layering cannot affect the diffusive motion. Although Eq. (4) is correct as written, the parameters n_1 and n_2 are random variables, and we must decide how these are to be incorporated into the equation for $\Gamma(n_1, n_2, \rho | \mu_1, \mu_2)$.

There are a number of ways to handle this problem, including, for small values of n , enumerating all the physically possible decompositions into components n_1 and n_2 . However, we will choose a more heuristic approach that appears to yield a useful approximation. Observe first that a photon reflected back to the surface has either never reached the inner layer, in which case $n_2 = 0$, or it has reached the inner layer at least once. We assume that a photon that has reached the inner layer has done so exactly once. The probability of repeated interchanges of sojourns within the two regions will be small because otherwise a larger number of steps will have been taken than in the case of a single sojourn. This, however, increases the probability of absorption, thereby decreasing the likelihood of having such a random walk in place of the single sojourn case. We therefore neglect the contribution of paths with multiple sojourns in our derivation of an approximation. The assumption of a single sojourn allows us to speak of two sojourns in the upper layer, the first in which the photon travels downward from the upper surface and the second in which the photon travels back toward the surface. By the assumed symmetry of the random walk, the average number of steps in each of these sojourns will be equal, so that for a given value of n_1 the average number of steps per sojourn is equal to $n_1/2$. In the formulas to follow we will replace the random variable n_1 by a deterministic parameter m . We can determine the dependence of m on the model parameters D and μ_1 by a scaling argument, and the remaining coefficient is calculated by solving several representative cases and noticing that the coefficient that leads to good agreement with the simulated data is approximately constant for all cases.

The probability for being at the interface at depth D at step $n_1/2$ can approximately be taken equal to

$$Q_{n_1/2}(D) \approx \sqrt{\frac{3}{\pi n_1}} \exp\left(-\frac{3}{n_1} D^2 - \frac{\mu_1 n_1}{2}\right), \quad (5)$$

since only a single layer is being traversed. Because the

exponential term dominates the behavior of $Q_{n_1/2}(D)$ considered as a function of n_1 , we can to a good approximation maximize the value of this function by minimizing the value of the exponent. This procedure implies that the minimizing value of n_1 is proportional to $D/\sqrt{\mu_1}$. In fact we have found that good agreement with simulated data is found when we set

$$n_1 = m = 5.2D/\sqrt{\mu_1}. \quad (6)$$

This value will be substituted for n_1 in our calculation of the intensity profile. The intensity profile, summed over all numbers of steps, will be denoted by $\Gamma(\rho)$, where $\Gamma(\rho)d\rho$ can be interpreted as the probability that a photon exits the surface at a distance between ρ and $\rho + d\rho$. The function $\Gamma(\rho)$ is calculated as the sum of two terms, both of which can be expressed in terms of the function $\Gamma(n_1, n_2, \rho | \mu_1, \mu_2)$:

$$\Gamma(\rho) \approx \sum_{n=1}^m \Gamma(n, 0, \rho | \mu_1, 0) + \sum_{n=m+1}^{\infty} \Gamma(m, n - m, \rho | \mu_1, \mu_2). \quad (7)$$

The first sum on the right-hand side represents the contribution from sojourns in the upper layer in which the number of steps is still insufficient to allow the reemitted photon to reach the lower one, while the second sum gives the contribution from all remaining steps.

The expression for $\Gamma(\rho)$ given in Eq. (7) can be approximated, as in Ref. 1, by converting the sums to integrals. In this way we find as an approximation for the surface intensity profile

$$\Gamma(\rho) \approx \int_0^m \Gamma(n, 0, \rho | \mu_1, 0) dn + \int_m^{\infty} \Gamma(m, n - m, \rho | \mu_1, \mu_2) dn. \quad (8)$$

The integrals can be calculated in terms of an infinite series of Bessel functions, but for present purposes we evaluate the integrals approximately by expanding the integrands around their respective maxima when ρ is assumed to exceed 1. The first integrand, for example, can be written in terms of the function that is correct for the single layer problem:

$$\Gamma(n, 0, \rho | \mu_1, 0) = \Gamma(n, \rho | \mu_1), \quad (9)$$

where the expression for $\Gamma(n, \rho | \mu_1)$ is given in Eqs. (2) and (3). The governing factor for the asymptotic (in m) evaluation of the first integral in Eq. (8) is contained in the exponential terms, and we will accordingly only maximize these, ignoring the term in $n^{-3/2}$ which can be shown to lead to correction terms smaller than those retained. Because of the term $[1 - \exp(-6/n)]$ there are two terms in $\Gamma(n, 0, \rho | \mu_1, 0)$ considered as a function of n . In the first of these, i.e., in the term

$$\exp\left(-\frac{3\rho^2}{2n} - \mu_1 n\right)$$

the maximum value of the exponent occurs when n has the value

$$n_{m_1} = \rho \sqrt{\frac{3}{2\mu_1}}, \quad (10)$$

while in the second the maximum occurs at

$$n_{m_2} = \sqrt{\frac{3\rho^2 + 6}{2\mu_1}}. \quad (11)$$

Of these, the principal contributor to the value of the integral is n_{m_1} , since the additional factor of 6 in the numerator of Eq. (11) insures that $\exp(-\mu_1 n)$ cuts down its contribution to a greater extent. Hence we will assume that to a first approximation only the first term determines the asymptotic form of the first integral in Eq. (8). Accordingly, we find that the maximum contribution to the integral will come from the interior of the interval $(0, m)$ whenever $\rho < 2D$ and from the neighborhood of the upper limit m when $\rho \geq 2D$. A similar calculation allows us to find an approximate value for the second integral in Eq. (8).

On performing all the approximate integrations and collecting all the resulting terms, we find

$$\Gamma(\rho) \approx \frac{1}{4\pi\rho^2} \{ \sqrt{6\mu_1} \exp(-\rho\sqrt{6\mu_1}) + \sqrt{6\mu_2} \exp[-\rho\sqrt{6\mu_2} - m(\mu_1 - \mu_2)] \}. \quad (12)$$

This expression reduces to the single layer result given in Ref. 1 when $\mu_1 = \mu_2$. Some typical curves of $\log_{10}\Gamma(\rho)$ derived by the simulation techniques described in Ref. 2 are shown in Fig. 2. They indicate that the two-layer configuration is reflected in the transition between the behavior of $\Gamma(\rho)$ in the small and large ρ regimes. This dichotomous behavior can also be found in the approximation of Eq. (12) in the specification of the regimes in which one or the other of the two terms are dominant. To see this, let us, for example, examine the conditions in which the first term in Eq. (12) is the dominant one, i.e.,

$$\sqrt{6\mu_1} \exp[-\rho\sqrt{6\mu_1}] > \sqrt{6\mu_2} \exp[-\rho\sqrt{6\mu_2} - m(\mu_1 - \mu_2)]. \quad (13)$$

The principal factor determining this regime is the relation between the two exponentials. Reduction of the inequality implies that the first term is the larger one when

$$\rho < \frac{5.2}{\sqrt{6}} D \left(1 + \sqrt{\frac{\mu_2}{\mu_1}} \right), \quad (14)$$

where we have used the value of m given in Eq. (6). This relation implies that the measured intensity close to the point of entry of the laser beam (ρ small) is mainly due to photons not reaching the lower layer. As D increases the extent of this single layer regime also increases, as is evident from Eq. (14) and as one might expect on intuitive grounds. The behavior of the intensity profile for large ρ has a significant dependence on μ_2 because most of such photons will have traveled through the lower layer. Some typical curves generated from Eq. (12) are shown in Fig. 2, where they are compared to the results of simulation. The figure for the somewhat realistic set of parameters $\mu_1 = 0.4$ and $\mu_2 = 0.1$ indicate that from an experimental point of view it might be extremely difficult to estimate the depth of anything other than an extremely thin layer of pigment from surface measurements with instrumental noise. This is suggested, for example, by the point

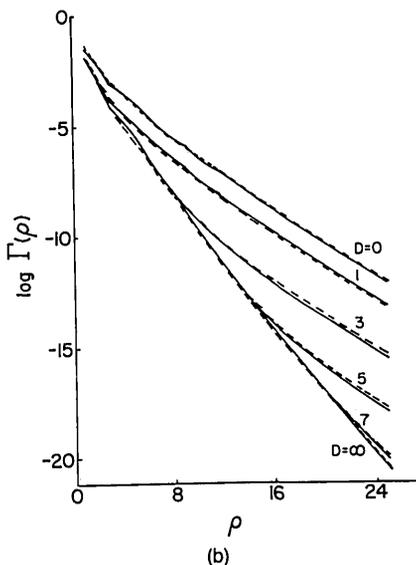
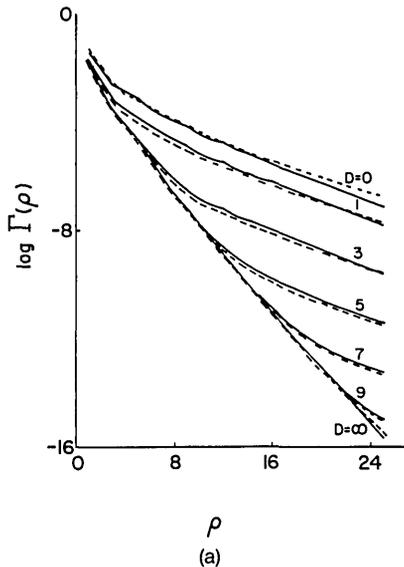


Fig. 2. Graphs of the exact and approximate values of the function $\log \Gamma(\rho)$ for $\mu_1 > \mu_2$ for different values of the layer thickness D . The approximations calculated using Eq. (12) are indicated by dashed lines, while those found by the exact enumeration method³ are given as solid lines. The parameters used for these calculations are (a) $\mu_1 = 0.2, \mu_2 = 0.01$ and (b) $\mu_1 = 0.4, \mu_2 = 0.1$.

at which the curve for $D = 3$ separates from that for $D = \infty$, which occurs at a value of $\Gamma(\rho) < 10^{-8}$.

The same technique used to derive an approximate form for $\Gamma(\rho)$ can also be used to generate an approximation to the expected path length of photons that emerge at $z = 0$ at a distance ρ from the point at which the laser beam enters the tissue. This quantity, which was calculated for an unlayered medium in Ref. 1, will be denoted by $\langle n|\rho \rangle$. It can be represented, in analogy to our calculation of $\Gamma(\rho)$, by the approximation

$$\langle n|\rho \rangle \approx \left[\int_0^m n \Gamma(n, 0, \rho | \mu_1, 0) dn + \int_m^\infty n \Gamma(m, n - m, \rho | \mu_1, \mu_2) dn \right] / \Gamma(\rho). \quad (15)$$

The evaluation of the integrals appearing in this equa-

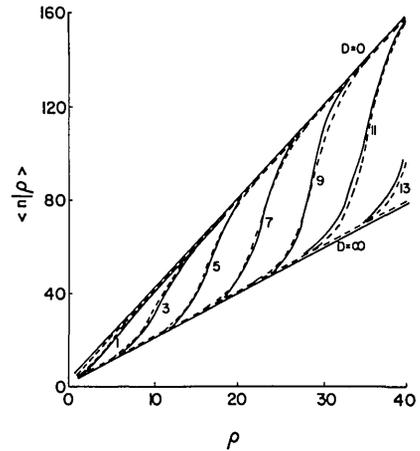


Fig. 3. Comparison of values of $\langle n|\rho \rangle$ as a function of ρ calculated by the exact enumeration method³ (solid lines) and the approximation of this paper (dashed lines) for $\mu_1 = 0.4$ and $\mu_2 = 0.1$ for several values of the thickness.

tion is almost identical to the evaluation of those in our calculation of $\Gamma(\rho)$. The resulting value of $\langle n|\rho \rangle$ is thereby found to have the form

$$\langle n|\rho \rangle \approx \frac{3\rho}{\sqrt{6\mu_1}} \left[\frac{1 + \sqrt{\frac{\mu_1}{\mu_2}} A(\rho)}{1 + A(\rho)} \right], \quad (16)$$

in which $A(\rho)$ is the function

$$A(\rho) = \sqrt{\frac{\mu_2}{\mu_1}} \exp[\rho(\sqrt{6\mu_1} - \sqrt{6\mu_2}) - m(\mu_1 - \mu_2)]. \quad (17)$$

This function has appeared implicitly in our calculation of $\Gamma(\rho)$ and has the limiting properties

$$A(\rho) \rightarrow 0 \text{ when } \rho \ll (5.2/\sqrt{6})D(1 + \sqrt{\mu_2/\mu_1}), \quad (18a)$$

$$A(\rho) \rightarrow \infty \text{ when } \rho \gg (5.2/\sqrt{6})D(1 + \sqrt{\mu_2/\mu_1}). \quad (18b)$$

The condition in Eq. (18a) corresponds to photon migration in a single medium with absorptivity equal to μ_1 and that in Eq. (18b) to migration in a single medium with absorptivity μ_2 as may be inferred from Eq. (16). Figure 3 compares simulated curves of $\langle n|\rho \rangle$ with those calculated from our present approximation. The breakpoint between the two regimes in ρ is evident also in these curves.

B. Pigmented Lower Layer ($\mu_2 > \mu_1$)

The case of a pigmented lower layer may be considered as a highly idealized model for an imaging system in which one wants to learn something about the presence of possible inhomogeneities within the medium from intensity measurements. Figure 4 shows curves of $\log_{10} \Gamma(\rho)$ as a function of ρ for this case. There is no hint in these curves of two regimes attributable to the two absorptivities, and in fact the curves resemble those found for photon migration in a homogeneous medium.¹ It is relatively simple to discuss the phenomenology for this case. Most of the photons prefer to remain in the upper layer and may exit from the upper surface without ever having traveled in the lower, more

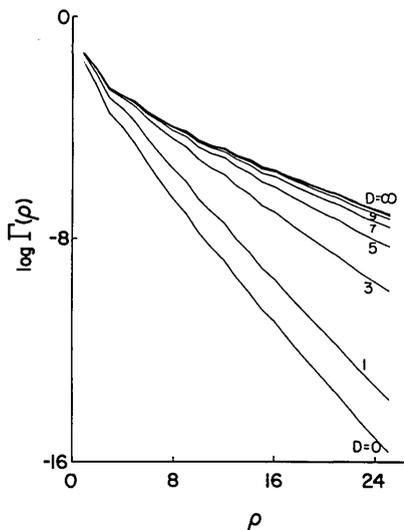


Fig. 4. Curves of $\log \Gamma(\rho)$ as a function of ρ obtained by the exact enumeration method³ for $\mu_1 = 0.01$ and $\mu_2 = 0.2$. The results closely resemble those given in Ref. 1 for a homogeneous medium.

absorptive, layer. Therefore, when D can be regarded as having a significant width, the intensity at the upper surface will be mainly derived from the class of photons which has not traveled through the lower layer to any great extent.

Examination of similar curves for $\langle n|\rho \rangle$ suggests that the parameters characterizing photon migration in media with a more heavily pigmented inner layer may be approximated by assuming that the migration occurs in a homogeneous medium with an effective absorptivity $\langle \mu \rangle$ representing an average over the two-component absorptivities. A heuristic argument⁴ suggests that $\langle \mu \rangle$ can be approximated by an expression of the form

$$\langle \mu \rangle = \frac{\alpha D^2 \mu_1 + \mu_2}{\alpha D^2 + 1}, \quad (19)$$

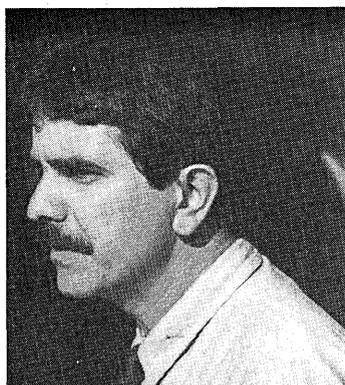
where α is a constant. This approximation has been fitted to some sets of simulated data, the results suggesting that α is of the order of μ_2 . However, since in biomedical applications one would hardly ever be in a position to know whether a given medium is homogeneous or composite from surface measurements alone, we have not examined this case further.

The practicability of being able to measure physiological parameters using these techniques would depend on which ones were of interest. If, for example, the absorption coefficients are known and the thickness of the upper layer is sought, our theoretical calculations indicate that a measurement of reflected intensity may indeed be a useful experimental approach provided that certain conditions are fulfilled. The first is that the absorptivity of the upper layer be greater than that of the lower one, and the second is that the thickness of the upper layer is not so large as to prevent penetration of the lower layer. If the absorption coefficients are themselves unknown, it would be quite difficult to infer both the coefficients and the layer thickness from experiments just described. The feasibility of parameter estimation when the lower layer is the more heavily pigmented one would seem to offer considerable experimental difficulties because of the absence of any qualitative differences between intensity profiles in the single and double layer cases.

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N. Peyghambarian, of the University of Arizona, photographed by W. J. Tomlinson, of Bellcore, at the International Conference on Optical Nonlinearity and Bistability of Semiconductors, East Berlin, in August 1988.