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Anderson localization in a correlated landscape near the band edge

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Abstract

We study the Anderson localization in a potential landscape with long-range power-law correlations in one dimension. We find that, for energies E close to the band edge of the unperturbed energy band, the presence of correlations makes the states more strongly localized, compared with the uncorrelated case, and a novel scaling of the localization length appears.

It is well known that the states of the one-dimensional Schrödinger equation in a random potential are usually localized (Borland 1963, Halperin 1965). These results have been found for random potentials without correlations between sites (Kappus and Wegner 1981, Gardner *et al.* 1984). It is generally believed that localization does not crucially depend on the local details or on short-range correlations.

Long-range power-law correlations of a local property appear to be common in nature. They have been found in biological systems such as the deoxyribonucleic acid sequences (Peng *et al.* 1992) and in physical systems, for example in porous media (Ischenko 1992, Vidales *et al.* 1996).

Here we study the effects of long-range spatial correlations in the random potentials on the localization properties of the states for energies close to the band edge of the unperturbed system and show that they modify the scaling of the localization length.

We study the tight-binding Anderson model with diagonal disorder on one-dimensional lattices:

$$\psi_{n+1} + \psi_{n-1} - 2\psi_n + \lambda V_n \psi_n = (E-2)\psi_n, \qquad (1)$$

with random V_n, distributed uniformly in the interval $[-1, 1]$ and λ being a positive constant, which describes the amplitude of the disorder. The energy E is a fixed parameter and we choose the local potentials to obey the correlation function $C(\ell)$ with a long-range power-law behaviour of the form (Makse *et al.* 1996)

$$C(\ell) = \langle V_n V_{n+\ell} \rangle \propto (1+\ell^2)^{-\gamma/2} \propto \ell^{-\gamma} \qquad \text{for} \qquad \ell \gg 1. \qquad (2)$$

Equation (1) can be written in terms of 2×2 matrices:

$$\hat{M}_n \begin{bmatrix} \psi_n \\ \psi_{n-1} \end{bmatrix} = \begin{bmatrix} \psi_{n+1} \\ \psi_n \end{bmatrix} \qquad \text{with} \qquad \hat{M}_n = \begin{bmatrix} E - \lambda V_n & -1 \\ 1 & 0 \end{bmatrix} \qquad (3)$$

The localization length can be defined by (Derrida and Gardner 1984, Kramer and MacKinnon 1993)

$$\Lambda(E,\lambda)^{-1} = \lim_{N\to\infty}\left[\frac{1}{N}\log\left(\frac{\psi_N}{\psi_0}\right)\right] \tag{4}$$

where $\Lambda(E,\lambda)^{-1}$ measures the exponential increases in ψ_n.

The ratio ψ_N/ψ_0 in the limit $N\to\infty$ is related to the eigenvalues of the product matrix (Derrida *et al.* 1987)

$$\hat{M}^N = \prod_{n=1}^{N}\hat{M}_n \qquad \text{with} \qquad \hat{M}^N\begin{bmatrix}\psi_1\\ \psi_0\end{bmatrix} = \begin{bmatrix}\psi_{N+1}\\ \psi_N\end{bmatrix}. \tag{5}$$

At the band edge $E = 2$, where the relevant length scales are large, one can obtain the λ dependence of the localization length $\Lambda(E=2,\lambda)$ by the following space decimation procedure. For blocks of b sites one redefines the potential in equation (1) by the mean value over the block:

$$V_b \sim \sum_b V_i. \tag{6}$$

The long-range correlation described by equation (2) leads to the following scaling of the second moment of the potential:

$$\langle V^2\rangle \to \langle V_b^2\rangle \propto b^{2-\gamma}\langle V^2\rangle. \tag{7}$$

Since space is rescaled by a factor b, the expression

$$\psi_{n+1} + \psi_{n-1} - 2\psi_n \tag{9}$$

transforms into

$$b^{-1}(\psi_{n+1}^{(b)} + \psi_{n-1}^{(b)} - 2\psi_n^{(b)}). \tag{10}$$

The next stage is transforming the equation for the block variables to a form identical with the original equation (equation (1)). We obtain this by transforming $\lambda \to \lambda_b = b\lambda$ and find that

$$\lambda_b^2\langle V_b^2\rangle \sim b^{4-\gamma}\lambda^2\langle V^2\rangle. \tag{11}$$

Assuming a power-law singularity for Λ at $E = 2$ given by

$$\Lambda(E=2,\lambda) \sim (\lambda^2\langle V^2\rangle)^{-y} \tag{12}$$

and with

$$\Lambda \to \Lambda_b \propto \frac{\Lambda}{b} \tag{13}$$

we derive using equations (11) and (13)

$$y = \frac{1}{4-\gamma}. \tag{14}$$

Note that for $\gamma \geqslant 1$

$$\langle V_b^2\rangle \propto \langle V^2\rangle \tag{15}$$

so that one recovers the exponent $y = \frac{1}{3}$, which was derived earlier for random systems (Derrida and Gardner 1984). Close to $E = 2$, the above scaling can be generalized using (Russ *et al.* 1997)

$$\frac{\Lambda(E, \lambda)}{\Lambda(E = 2, \lambda)} = f(x), \tag{16}$$

where

$$x = \frac{(2 - E)}{(\lambda^2 \langle V^2 \rangle)^{2/(4-\gamma)}}, \tag{17}$$

and $f(0) = $ constant.

Our scaling theory is supported by numerical calculations. We generate correlated one-dimensional sequences of size (equal to number of particles) $N = 2^{17}$ using the Makse *et al.* (1996) method and compute the localization lengths $\Lambda(E, \lambda)$, associated with the eigenvalues of the product of random matrices. In figure 1 we show the scaling of the localization length Λ with $\lambda \langle V^2 \rangle^{1/2}$ in the case $E = 2$ (equation (12)) on a double-logarithmic plot for randomly distributed chains and for correlated chains with correlation exponents $\gamma = 0.1$ and 0.9. The resulting curves are straight

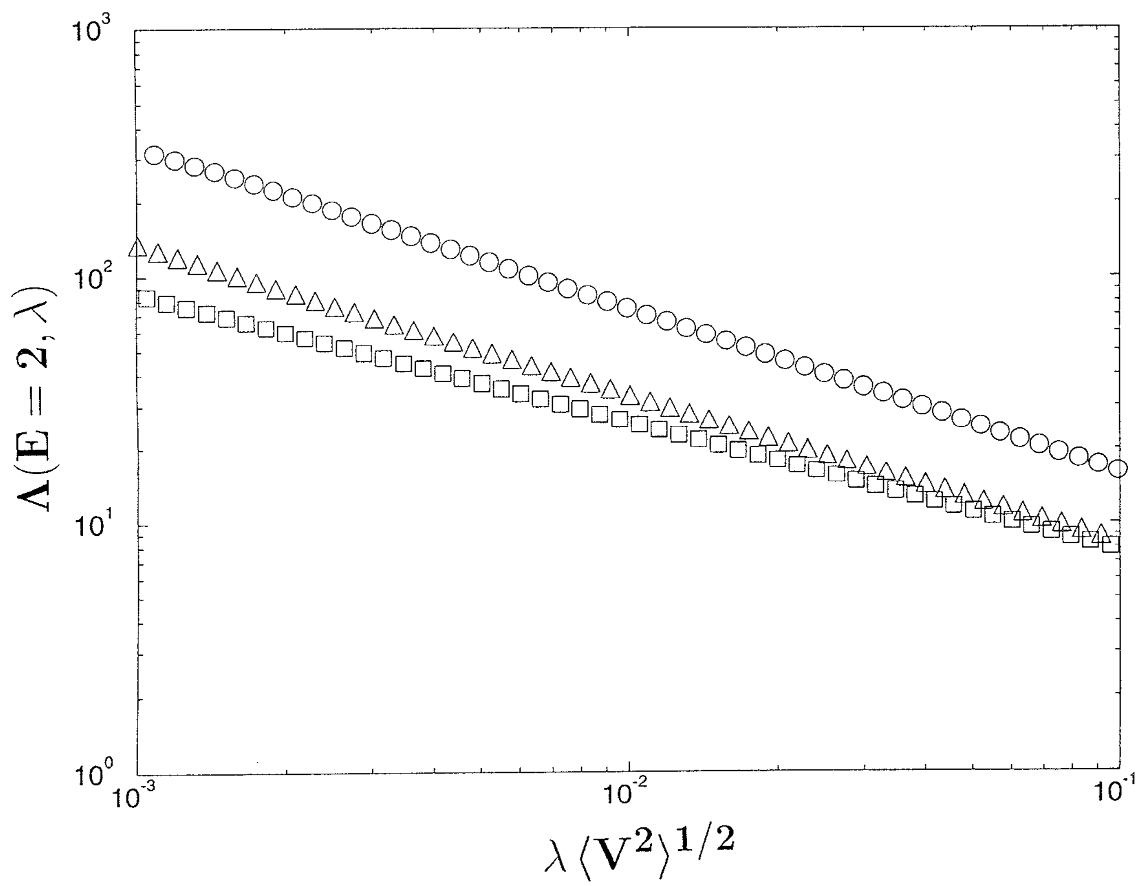

Figure 1. We plot Λ against $\lambda \langle V^2 \rangle^{1/2}$ in a double-logarithmic scale for $E = 2$ for different values of the correlation exponent $\gamma = 0.1$ (□), 0.9 (△) and for randomly distributed chains (○). The averages over $1/\Lambda$ were performed over 100 systems of length 2^{17}. The resulting curves are straight lines with the slopes close to $-2/3$ for the random case and close to -0.51 for $\gamma = 0.1$ and to -0.65 for $\gamma = 0.9$ in the correlated cases, in accordance with the theory.

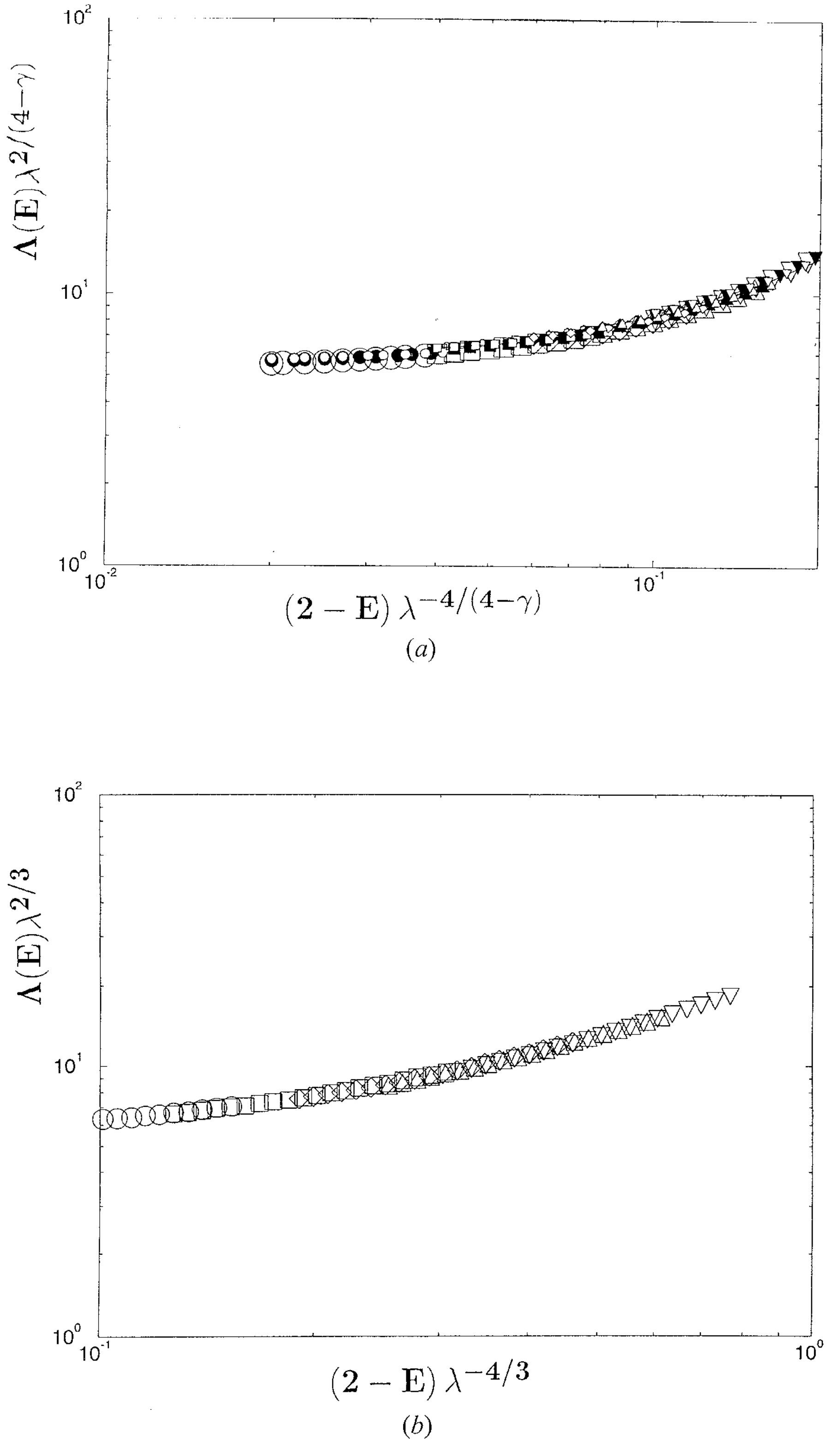

Figure 2. Plot of the scaling function for $1.9 \leqslant E < 2$ (cf. equation (16) and (17)). The average over $1/\Lambda$ was performed over 100 systems of length 2^{17}. The different symbols are for different energies $E = 1.98$ (○, ●, ◯), $E = 1.96$ (□, ■, ☐), $E = 1.94$ (◇, ◆, ◇), $E = 1.92$ (△, ▲, △), and $E = 1.90$ (▽, ▼, ▽). We varied the amplitude of disorder λ in the interval [0.5, 1.0] in the correlated case and in [0.2, 0.4] in the random case, corresponding to about the same range of $\lambda^2\langle V^2\rangle$. In (a), the correlated case, we plot $\Lambda(E)\lambda^{2/(4-\gamma)}$ against $(2-E)/\lambda^{4/(4-\gamma)}$: (◯), (☐), (◇), (△), (▽), data for $\gamma = 0.1$; (●), (■), (◆), (▲), (▼), data for $\gamma = 0.3$; (○), (□), (◇), (△), (▽), data for $\gamma = 0.5$. In (b), the random case, we plot $\Lambda(E)\lambda^{2/3}$ against $(2-E)/\lambda^{4/3}$.

lines with slopes close to $-2/3$ for the random case and close to $-2/(4-\gamma)$ in the correlated cases, in accordance with the theory. These results mean that, the more correlated the random potential, the stronger is the localization. Thus states of random chains are less localized than those of correlated chains at the band edge. It is clear from equations (7) and (12) that at the band edge the presence of long-range correlations makes the localization more pronounced than in the random case. The main reason for this effect is the strong enhancement of long-wavelength fluctuations of the potential, responsible for localization near $E = 2$.

Figure 2 shows the scaling for $E < 2$. In figure 2 (*a*), the correlated case, we plot $\Lambda\lambda^{2/(4-\gamma)}$ against $(2-E)/\lambda^{4/(4-\gamma)}$ for $\gamma = 0.1$, 0.3 and 0.5. In figure 2 (*b*), the random case, we plot $\Lambda\lambda^{2/3}$ against $(2-E)/\lambda^{4/3}$. The data collapse supports the scaling assumptions (16) and (17). We find data collapse for $E \leqslant 1.9$, $\lambda \geqslant 0.5$, in the correlated case and a gradually breakdown of the scaling assumption if we leave this regime.

In summary we have shown that, close to the edge of the unperturbed energy band, the presence of correlations makes the states more strongly localized. This counterintuitive effect is related to the statistics of sums of correlated random variables. We find a new scaling for the localization length by using decimation arguments, which fit very well with the numerical results.

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