International Journal of Bifurcation and Chaos, Vol. 7, No. 4 (1997) 837–838

ANALYTICAL SOLUTION OF THE LOGISTIC EQUATION

S. RABINOVICH*, G. BERKOLAIKO*,†, S. BULDYREV‡, A. SHEHTER*, S. HAVLIN*
*Department of Physics, Bar-Ilan University, 52900 Ramat-Gan, ISRAEL
†Department of Mathematics, Voronezh University, 394693 Voronezh, RUSSIA
‡Department of Physics, Boston University, Boston, MA 02215, U.S.A.

Received August 21, 1995; Revised January 26, 1996

An analytical solution for the well-known quadratic recursion, also known as the *logistic map*, is presented. Our derivation makes use of the analogy between this equation and Galton–Watson processes. The latter can be treated in a linear way using the transfer matrix technique. The solution is represented as a power of an appropriate transfer matrix.

The logistic map is one of the famous equations in physics and biology. Therefore, it needs no special introduction.[1] In this letter we construct an analytical solution for the logistic recursion

$$f_{n+1} = \lambda f_n(1 - f_n) \equiv \mathbf{\Lambda}(f_n) \quad \text{with} \quad f_0 \equiv f\,. \tag{1}$$

We first consider the following auxiliary problem. It is known as a simple Galton–Watson probabilistic branching process.[2] Let a particle at each time step give birth to another identical one with the probability p or just survive with the probability q ($q = 1 - p$). The quantity under question is the probability, $P_{n,k}$, to find an (n, k)-state, i.e. k particles after n time steps (nth *generation*). The initial state of the system is exactly one particle: $P_{0,1} = 1$. This problem can be treated in two ways.

The first one is to write down a relation for $P_{n,k}$,

$$P_{n+1,k} = qP_{n,k} + p\sum_m P_{n,m}P_{n,k-m}\,. \tag{2}$$

This formula can be understood in the following way. We can get k particles in $(n+1)$ generation if at the first time step:

- the particle just survives (with the probability q) and then the population grows to k particles in n time steps (with the probability $P_{n,k}$); or
- the particle gives birth to one more (with the probability p) and then one of the particles gives m descendants after n time steps while another one gives $k - m$ descendants.

By switching to a generating function $\tilde{P}_n(x) = \sum_k P_{n,k}x^k$, Eq. (2) transforms to

$$\tilde{P}_{n+1}(x) = q\tilde{P}_n(x) + p[\tilde{P}_n(x)]^2\,. \tag{3}$$

Note, that the initial condition implies $\tilde{P}_0(x) = x$.

While being established for $p + q = 1$, expression (3) is defined for *any* p and q. In particular, substituting in Eq. (3) $q = \lambda$, $p = -\lambda$, one has

$$\tilde{P}_{n+1}(x) = \mathbf{\Lambda}^{n+1}\left(\tilde{P}_0(x)\right) = \mathbf{\Lambda}^{n+1}(x)\,, \tag{4}$$

where $\mathbf{\Lambda}^{n+1}$ is $(n+1)$th iteration of the logistic map.

The second way to solve our auxiliary problem is to calculate $P_{n,k}$ directly. Let us suppose that we have $k - m$ particles in nth generation (with the

[1] For a good review of the problem and relevant references see, for example, H.-O. Peitgen, H. Jürgens, D. Saupe, *Chaos and Fractals* (Springer, New York, 1992)

[2] S. Asmussen, H. Hering, *Branching Processes* (Birkhäuser, Boston, 1983)

probability $P_{n,\,k-m}$). We shall find k particles in $(n+1)$th generation if $k-2m$ particles just survived and the rest m particles gave birth to another m new particles. The probability of a single event is $p^m q^{k-2m}$ and its weight is $\binom{k-m}{m}$. Thus, it is obvious that

$$P_{n+1,\,k} = \sum_{m=0}^{[k/2]} \binom{k-m}{m} p^m q^{k-2m} P_{n,\,k-m}\,, \qquad (5)$$

where $[k/2]$ stands for the integer part of $k/2$ and $\binom{n}{m}$ is the binomial coefficient. The latter equation can be written in a matrix form as a transfer matrix relation:

$$|\mathbf{P}_{n+1}\rangle = \mathbf{T}|\mathbf{P}_n\rangle \qquad (6)$$

where

$$T_{j,\,k} = \binom{k}{j-k} p^{j-k} q^{2k-j} \qquad \text{and}$$

$$\langle \mathbf{P}_n| = (P_{n,\,1}, P_{n,\,2}, \ldots)\,.$$

The solution of Eq. (6) is simply

$$|\mathbf{P}_n\rangle = \mathbf{T}^n|\mathbf{P}_0\rangle\,, \qquad (7)$$

where $\langle \mathbf{P}_0| = (1, 0, \ldots)$ is determined from the initial condition.

To complete the derivation one should find the generating function $\tilde{P}_n(x)$ from Eq. (7). This is done by multiplying the ket-vector $|\mathbf{P}_n\rangle$ by the bra-vector of powers of x, $\langle \mathbf{x}| = (x, x^2, x^3, \ldots)$. Thus,

$$\tilde{P}_n(x) = \langle \mathbf{x}|\mathbf{T}^n|\mathbf{P}_0\rangle\,. \qquad (8)$$

It is interesting to note that the first approach leads to a nonlinear relation, Eq. (2), while the second approach to the same problem leads to a solvable linear recursion, Eq. (5).

Now we observe that the solution of the auxiliary problem can be related to the logistic map (1). Comparing Eq. (4) with Eq. (8) one obtains

$$\mathbf{\Lambda}^n(x) = \langle \mathbf{x}|\mathbf{T}^n|\mathbf{P}_0\rangle\,. \qquad (9)$$

It is important to note, that no restrictions on x were imposed. Then, inserting in Eq. (9) the initial condition, $x = f$, and recollecting, that $f_n = \mathbf{\Lambda}^n(f)$, one obtains

$$f_n = \langle \mathbf{f}|\mathbf{T}^n|\mathbf{P}_0\rangle\,, \qquad (10)$$

where $\mathbf{T}$ is defined by

$$T_{j,\,k} = (-1)^{j-k} \binom{k}{j-k} \lambda^k\,, \qquad (11)$$

$\langle \mathbf{f}| = (f, f^2, f^3, \ldots)$ and $\langle \mathbf{P}_0| = (1, 0, 0, \ldots)$. Eq. (10) is the solution of the logistic recursion (1).

This research was supported in part by the Israel Science Foundation.