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LETTER TO THE EDITOR

Diffusion in the presence of random fields in $d \geqslant 2$ dimensions

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Abstract. We study diffusion in the presence of random fields in d-dimensional systems. We find an exact upper bound for the mean-square displacement $\langle R^2 \rangle \leqslant n^{2\nu}$ where ν is the self-avoiding walk end-to-end exponent. We also present numerical results in two dimensions which indicate normal diffusion for $d \geqslant 2$. This is consistent with the above exact bound.

Diffusion in the presence of random fields has been studied by several authors [1–5]. Sinai [1] found that for a symmetrical distribution of local random fields the diffusion in one-dimensional systems is anomalous, characterised by a mean-square displacement $\langle x^2 \rangle$ which scales as

$$\langle x^2 \rangle \sim \log^4 n \tag{1}$$

where n is the number of steps.

The problem of diffusion in the presence of random fields in $d = 3$ dimensions and in percolation systems has been studied recently by Pandey [5, 6] using Monte Carlo simulations. On the basis of his numerical data Pandey speculated that in $d = 3$ dimensions the mean-square displacement $\langle R^2 \rangle$ increases as a function of time n faster than that of diffusion and asymptotically approaches a drift, i.e. $\langle R^2 \rangle \sim n^2$.

In this letter we present theoretical arguments that $\langle R^2 \rangle \leqslant n^{2\nu}$ where ν is the self-avoiding walk exponent, $\nu \simeq 3/(d+2)$ for $1 \leqslant d \leqslant 4$. We also present numerical simulations of diffusion in the presence of random fields in $d = 2$ dimensions using the exact enumeration method [7]. The numerical results suggests that regular diffusion already occurs for $d = 2$, i.e. $\langle R^2 \rangle \sim n$, and that random fields have no influence for $d \geqslant 2$.

In the following we consider the general problem, where the probabilities of a walker to go from point i to its neighbours $\{j\}$ are positive random numbers $P_{i \to j}$ obeying $\Sigma_{j=1}^{z} P_{i \to j} = 1$, where z is the number of nearest neighbours. It is assumed that $p_{i \to j}$ and $P_{i' \to j'}$ are independent random variables for $i \neq i'$. The random field problems described by Pandey or in the last paragraphs of this letter are special cases of this general problem.

The mean-square distance from the origin of a random walker, after n steps, is given by

$$\langle R^2(n) \rangle = \sum_{C_n} R^2(C_n)\, W(C_n) \tag{2}$$

where the sum over C_n is over all walks of n steps starting at a given point and $W(C_n)$ is the probability of such a walk

$$W(C_n)=\prod_{i=1}^{n} P_{i\to i+1}(C_n) \tag{3}$$

where $P_{i\to i+1}(C_n)$ is the probability to go from the ith site to the $(i+1)$th site along the walk C_n. (Probabilities with different i appearing in the product may correspond to the same site.) The mean-square distance $\langle R^2(n)\rangle$ is a functional of the local probabilities $P_{i\to j}$. We are interested in the quenched average of that quantity, namely in the average over probability configurations of $\langle R^2(n)\rangle$ and we denote it by $\overline{\langle R^2(n)\rangle}$. Now

$$\overline{\langle R^2(n)\rangle}=\sum_{C_n} R^2(C_n)\overline{W}(C_n). \tag{4}$$

Let us break the sum on the right-hand side of (4) into a sum over the self-avoiding walks and all the rest:

$$\begin{aligned}\sum_{C_n} R^2(C_n)\overline{W}(C_n)&\\ &=\sum_{C_n}{}_{\mathrm{SAW}}\, R^2(C_n)\overline{W}(C_n)+\sum_{C_n}{}' R^2(C_n)\overline{W}(C_n)\\ &=\left[\sum_{C_n}{}_{\mathrm{SAW}}\, R^2(C_n)\overline{W}(C_n)\left(\sum_{C_n}{}_{\mathrm{SAW}}\,\overline{W}(C_n)\right)^{-1}\right]\left(\sum_{C_n}{}_{\mathrm{SAW}}\,\overline{W}(C_n)\right)\\ &\quad+\left[\sum_{C_n}{}' R^2(C_n)\overline{W}(C_n)\left(\sum_{C_n}{}'\,\overline{W}(C_n)\right)^{-1}\right]\left(\sum_{C_n}{}'\,\overline{W}(C_n)\right).\end{aligned} \tag{5}$$

Denoting

$$\sum_{C_n}{}_{\mathrm{SAW}}\,\overline{W}(C_n)=P_{\mathrm{SAW}}^{(n)} \tag{6}$$

we see that

$$\overline{\langle R^2(n)\rangle}=\overline{\langle R^2(n)\rangle}_{\mathrm{SAW}}P_{\mathrm{SAW}}^{(n)}+\overline{\langle R^2(n)\rangle}'(1-P_{\mathrm{SAW}}^{(n)}) \tag{7}$$

where $\overline{\langle R^2(n)\rangle}_{\mathrm{SAW}}$ is the mean-square distance of the self-avoiding walks (SAW) alone and $\overline{\langle R^2(n)\rangle}'$ is the average over the non-self-avoiding walks. Clearly,

$$\overline{\langle R^2(n)\rangle}_{\mathrm{SAW}}\geqslant\overline{\langle R^2(n)\rangle}' \tag{8}$$

and since $\overline{\langle R^2(n)\rangle}$ is just a weighted average of the two quantities we conclude that

$$\overline{\langle R^2(n)\rangle}\leqslant\overline{\langle R^2(n)\rangle}_{\mathrm{SAW}}. \tag{9}$$

Consider now $\overline{\langle R^2(n)\rangle}_{\mathrm{SAW}}$, since the probability of a self-avoiding walk is a product of n different and independent probabilities, the average probability for such a walk is given by $\overline{W}(C_n)=(1/z)^n$. Therefore,

$$\begin{aligned}\overline{\langle R^2(n)\rangle}_{\mathrm{SAW}}&=\sum_{C_n}{}_{\mathrm{SAW}}\, R^2(C_n)(1/z)^n\left(\sum_{C_n}(1/z)^n\right)^{-1}\\ &=\sum_{C_n}{}_{\mathrm{SAW}}\, R^2(C_n)\left(\sum_{C_n}{}_{\mathrm{SAW}}\,1\right)^{-1}\end{aligned} \tag{10}$$

which is just the usual mean-square distance of self-avoiding walks characterised by the a power behaviour in n, so

$$\langle R^2(n)\rangle \leqslant Cn^{2\nu} \tag{11}$$

where 2ν is given by $2\nu \simeq 6/(d+2)$ for $d \leqslant 4$ and $2\nu = 1$ for $d > 4$ (see, e.g., [8]), so that in two dimensions $2\nu = 1.5$ and in three dimensions $2\nu = 1.2$.

This result suggests that the numerical results of Pandey [5], where in some cases he finds $\langle R^2(n)\rangle \propto n^2$, cannot represent the asymptotic behaviour.

Now it is obvious from inequality (11) that for dimensions higher than four the behaviour of $\langle R^2(n)\rangle$ as a function of n is that of usual diffusion without quenched random behaviour. In one dimension the diffusion in the presence of quenched random fields is anomalous [1]. The question is whether anomalous behaviour is to be expected for dimensions $1 < d \leqslant 4$. Obviously the answer to the above problem depends on the possible correlations between R^2 and the topological structure of the walks. If no such correlation exists or if it is weak enough, normal diffusion is to be expected. In that sense quenched disorder would have the same effect as annealed disorder, where it can be trivially shown that the behaviour is that of normal diffusion; see, e.g., the review articles [7, 9].

If anomalous diffusion takes place, it is expected to be more anomalous as the dimension is lowered. We present in the following paragraphs some numerical indications that in two dimensions the diffusion is normal, thus implying that this should also be the behaviour for any $d \geqslant 2$.

The model we study numerically here is similar to Pandey's model. We study a lattice in d-dimensional space where to each site is attached a local field which points randomly to one of the $z = 2d$ directions (in hypercubic lattices). The probabilities of the walker to step out from such a site are $(1+\varepsilon)/2d$ in the field direction and $(1-\varepsilon)/2d$ in the opposite direction. In all other directions the probability to step is $1/2d$.

In figure 1 we present results for the mean-square displacement on a square lattice in the presence of a random field with $\varepsilon = 0.5$. The simulations were performed on a

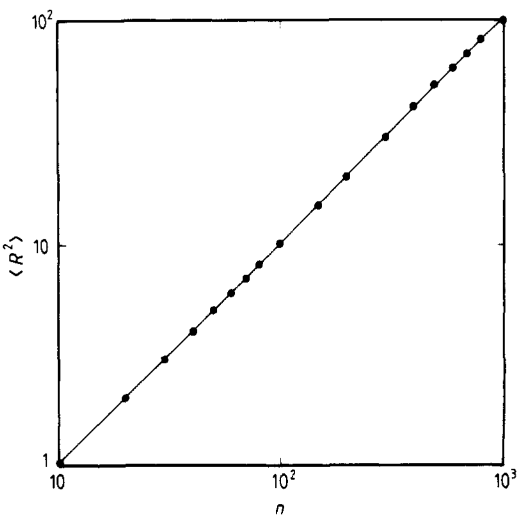

Figure 1. Plot of $\log\langle R^2\rangle$ as a function of $\log n$ for a random field $\varepsilon = 0.5$. The results were performed using the exact enumeration method [7] and averaging over 100 configurations.

lattice of 200×200 size using the exact enumeration method [7]. The figure shows that the values of $\langle R^2 \rangle$ are very close to the value of n. The slope in figure 1 is 1.0 ± 0.02 and similar results were obtained for different values of $\varepsilon (0 \leqslant \varepsilon < 1)$. This suggests that

$$\langle R^2 \rangle \simeq n \tag{12}$$

for any $0 \leqslant \varepsilon \leqslant 1$.

In summary, our theoretical and numerical results indicate that for $d \geqslant 2$ diffusion in the presence of symmetrical random fields is normal and is not affected by the presence of random fields.

References

[1] Sinai Ya 1982 *Theor. Prob. Appl.* **27** 256
[2] Kesten H 1986 *Physica* **138A** 299
[3] Derrida B and Pomeau Y 1982 *Phys. Rev. Lett.* **48** 627
[4] Roman H E, Bunde A and Havlin S 1986 *Preprint*
[5] Pandey R B 1986 *J. Phys. A: Math. Gen.* **19** 3925
[6] Pandey R B 1987 *J. Phys. A: Math. Gen.* **20** L337
[7] Havlin S and Ben-Avraham D 1987 *Adv. Phys.* **36** 695
[8] de Gennes P G 1979 *Scaling Concepts in Polymer Physics* (Ithaca, NJ: Cornell University Press)
[9] Haus J W and Kehr K W 1987 *Phys. Rep.* **150** 263