

## Optimal path in weak and strong disorder

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### Abstract

We generate optimal paths between two given sites on a lattice representing a disordered energy landscape by applying the Dijkstra algorithm. We study the geometrical and energetic scaling properties of the optimal path under two different energy distributions that yield the weak and strong disorder limits. Our numerical results, for both two and three dimensions, suggest that the optimal paths in weak disorder are in the same universality class as the directed polymers and in the strong disorder limit are fractals with exponents similar to that found by Cieplak et al. (Phys. Rev. Lett. 72 (1994) 2320; 76 (1996) 3754). © 1999 Published by Elsevier Science B.V. All rights reserved.

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Optimal paths have been much of interest recently. The optimal path can be defined as follows. Consider a  $d$ -dimensional lattice, where each bond is assigned with a random energy value taken from a given distribution. The optimal path between two sites is defined as the path on which the sum of the energies is minimal. This problem is of relevance to various fields, such as spin glasses [1], protein folding [2], paper rupture [3], and traveling salesman problem [4]. Though much effort has been devoted to study this problem, the general solution is still lacking. There exist two approaches developed recently to study this problem. Cieplak et al. [5] applied the max-flow algorithm for a two-dimensional energy landscape. Another approach is to restrict the path to be directed, that is, the path cannot turn backwards. This approach is the directed polymer problem which has been extensively studied in the past years, see e.g., [6–9].

In this manuscript, we adapt the *Dijkstra algorithm* from graph theory [10] for generating the optimal path on a lattice with randomly distributed non-negative energies assigned to the bonds [11]. This algorithm enables us to generate the optimal path

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between any two sites on the lattice, not restricted to directed paths. We study the geometrical and energetic properties of the optimal paths in  $d = 2$  and 3 dimensions using two different energy distributions. The first distribution of energies is a uniform distribution,  $P(E) = \text{const.}$  which gives the weak disorder limit, and the second distribution is  $P(E) \propto 1/E$  which generates the strong disorder limit as will be explained later. We calculate the scaling exponents for the width and the energy fluctuations of the optimal path in each case. For the distribution  $P(E) = \text{const.}$  we find that for both  $d = 2$  and 3 the exponents are very close to those of directed polymers suggesting that the non-directed optimal path (NDOP) is in the same universality class as directed polymer (DP). Our results are in agreement with those found by Cieplak et al. [5] for the two-dimensional case. This result indicates that in the case of uniformly distributed energies the NDOPs are self affine and overhangs do not play an important role in the geometry of NDOPs. For  $P(E) \propto 1/E$  we find that, for both  $d = 2$  and 3, the optimal paths are self similar and the exponents are very close to those found by Cieplak et al. [13] who applied an alternative algorithm.

In the case of the uniform distribution we simulate both DPs and NDOPs on a square lattice in the following way. Let  $x, y$  be the horizontal and vertical axes. We choose the origin to be the source site and study the optimal paths connecting it with all the sites on the line between  $[0, t]$  and  $[t, 0]$  for different values of  $t$ . In order to compare NDOP to DP we define the global optimal path as the minimal energy path among all the paths with the same value of  $t$ . In Fig. 1 we compare a configuration of DP and NDOP on the same disordered energy landscape. It is seen that in the NDOP only very few overhangs exist. To test the effect of the overhangs we calculate the mean end-to-end distance  $R$  of the global optimal path as a function of its length  $\ell$ . Our numerical results clearly indicate the asymptotic relation  $\ell \sim R$  showing that the NDOPs are self-affine [8]. We study several properties, such as the roughness exponent  $\xi$ , the energy fluctuation exponent  $\zeta$  for two and three dimensions, as well as the distribution of the endpoints of DP and NDOP (see Table 1 and Fig. 2). The above exponents are defined by the relations  $W \equiv \langle h^2 \rangle^{1/2} \sim t^\xi$  and  $\Delta E \equiv \langle (E - \langle E \rangle)^2 \rangle^{1/2} \sim t^\zeta$ . Here,  $h$  is the transverse fluctuation of the global optimal path which is distance between its endpoint and the line  $x = y$ ;  $E$  is the energy of the global optimal path which is the sum of all bond energies along the path. The average is taken over different realizations. The generalization to three dimensions is straightforward. We find that our results are independent of the distribution interval.

As for the strong disorder limit, where the sum of energies along the path is governed by one energy value, a uniform distribution is not suitable to reach this limit. Consider a lattice, and an energy interval of  $(0, A)$ , then most of the bonds will have a value which is of the order of  $A$  for every  $A > 0$ . Therefore, it will be impossible to connect two sites in the optimal way and have the optimal path energies behave in a strong disorder fashion. We therefore suggest an energy distribution  $P(\log E) = \text{const.}$  (i.e.  $P(E) \propto 1/E$ ) thus insuring that for different randomly generated integers will correspond two values that differ at least by an order of magnitude. Let us consider now a lattice and the

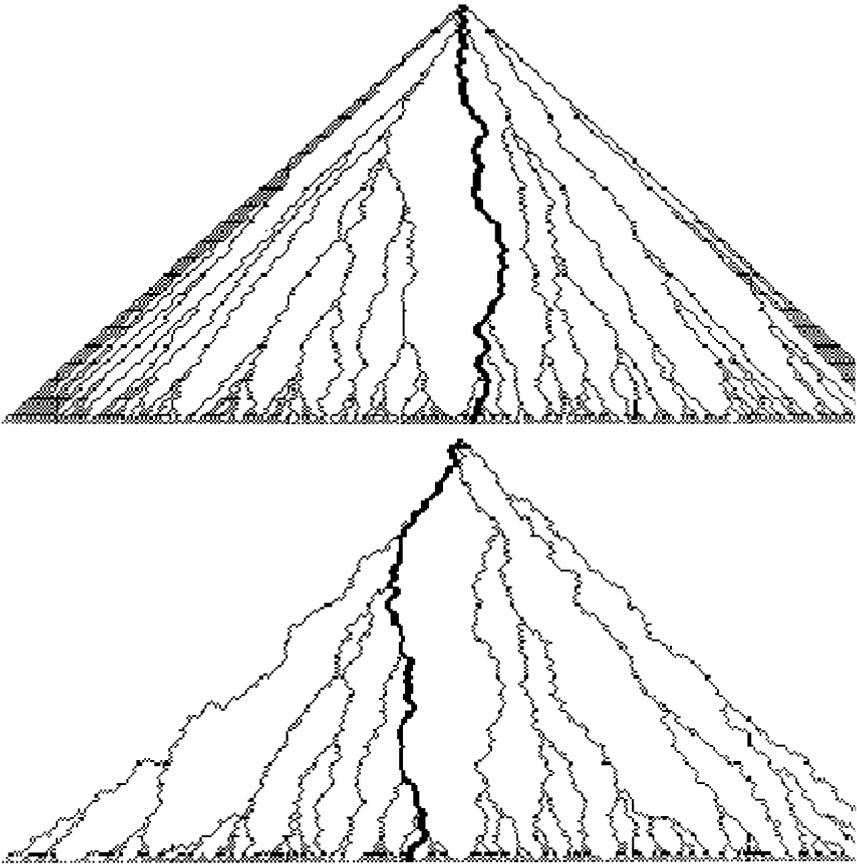


Fig. 1. The sets of all directed (the upper one) and non-directed (the lower one) optimal paths with  $t = 300$  obtained for the same realization of quenched randomness in the lattice. The global optimal path which is the minimal energy path among all the paths with the same  $t$  is shown by a thick line. In this particular case the directed and non-directed global optimal paths do not overlap. In other cases they might overlap significantly though the rest of tree looks somewhat different.

proposed distribution taking the interval to be  $(0, \infty)$  then for every finite lattice which has  $N$  bonds we are guaranteed to have  $N$  different values which differ from each other at least by an order of magnitude thus insuring the existence of the optimal path in the strong disorder limit. The effect of a cutoff,  $E_{\max}$ , on this distribution and the crossover from the strong disorder to the weak disorder behavior have been studied recently by Porto et al. [12]. Concentrating on the regime of the strong disorder, we study the scaling properties of the optimal paths. We find that  $d_{\text{opt}} = 1.22 \pm 0.02$  in  $d = 2$  and  $d_{\text{opt}} = 1.43 \pm 0.03$  in  $d = 3$  where  $d_{\text{opt}}$  is defined by  $\ell \sim R^{d_{\text{opt}}}$  and  $\ell$  is the path's length. These findings are in close agreement with the results obtained by Cieplak et al. [13–16] which are  $d_{\text{opt}} = 1.22 \pm 0.01$  for  $d = 2$  and  $d_{\text{opt}} = 1.42 \pm 0.02$  for  $d = 3$ .

Table 1

Width and energy fluctuation exponents of DP and NDOP in two and three dimensions using uniform distribution. The error bar were estimated from taking five ensembles of  $10^4$  configurations each for  $d = 3$  and 500 configurations each for  $d = 3$

|         | $d = 2$         |                 | $d = 3$         |                 |
|---------|-----------------|-----------------|-----------------|-----------------|
|         | DP              | NDOP            | DP              | NDOP            |
| $\xi$   | $0.66 \pm 0.02$ | $0.67 \pm 0.02$ | $0.60 \pm 0.05$ | $0.63 \pm 0.05$ |
| $\zeta$ | $0.32 \pm 0.02$ | $0.32 \pm 0.02$ | $0.19 \pm 0.07$ | $0.18 \pm 0.07$ |

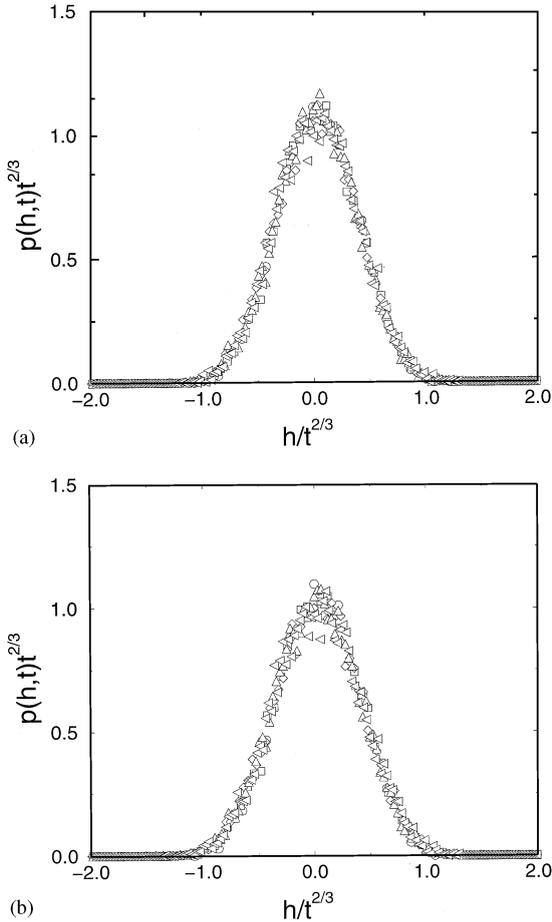


Fig. 2. Scaling of the distribution  $P(h, t)$  of the endpoints in (a) DP and (b) NDOP. The different symbols represent different values of  $t$  between  $t = 10$  and  $300$ .

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