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Loopless percolation clusters

Frank Tzschichholz
I. Institut für Theoretische Physik, Universität Hamburg, D-2000 Hamburg 36, Federal Republic of Germany

Armin Bunde and Shlomo Havlin
I. Institut für Theoretische Physik, Universität Hamburg, D-2000 Hamburg 36, Federal Republic of Germany and Department of Physics, Bar-Ilan University, Ramat-Gan, Israel

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We suggest a novel percolation model where the generated clusters are *loopless structures*. In this model sites are occupied with probability p and blocked with probability $1-p$, but those sites closing loops are considered as blocked sites with probability 1 ("green sites"). Using scaling analysis and numerical simulations in $d=2$, we find that the static exponents are the same as for conventional percolation. In contrast, the dynamical exponents such as the diffusion and resistance exponents are significantly different. This result supports the argument that in percolation the dynamical exponents are *independent* of the static exponents.

In recent years, the static and dynamical features of percolation systems have been studied extensively.[1-12] This is due to the fact that many phenomena in nature, ranging from the spreading of epidemics[13] and diffusion fronts[14] to the peculiar behavior of the resistance in disordered ionic conductors,[15] can be understood by percolation theory. While the static exponents characterizing the geometrical structure of percolation clusters (such as β, ν, and the fractal dimension d_f) are known exactly for two-dimensional percolation,[16] only numerical estimates exist for the resistance exponent $\tilde{\zeta}$ and the diffusion exponent d_w (see, e.g., Ref. 12). The question if there exists a relation between the dynamical exponents and the static exponents has been discussed extensively.[5,8,12,17-19]

The difficulties in studying theoretically the dynamical properties of percolation systems are mostly due to the complexity of the loops which appear in the clusters in all length scales. In order to better understand the effect of these loops on the geometrical and dynamical features, we introduce and study a model of correlated percolation in which the minimal number of sites needed to generate loops are taken away (green sites). The generated clusters are therefore loopless. We find that the static geometrical properties are not affected by blocking the green sites while the dynamical exponents are significantly modified, $\tilde{\zeta}=d_f/d_l$, where d_l is the chemical dimension[20] of the cluster.

To generate loopless percolation clusters we use an algorithm which is a modification of the Leath growth method.[21] In the first step, the origin of a lattice is occupied by a cluster site and its nearest-neighbor sites are occupied randomly with probability p and blocked with probability $1-p$. The empty nearest neighbors of the cluster sites form the perimeter sites. In the second step (which differs from Leath growth method), those sites closing loops are blocked with probability $c=1$. Sites that close loops are those perimeter sites that have two or more nearest-neighbor cluster sites. We call these blocked sites green sites. The remaining perimeter sites form the growth sites which then are occupied randomly with probability p or blocked with probability $1-p$. The procedure is continued in the next steps thus avoiding loops in all length scales. In this algorithm, in the lth step, cluster sites in the lth chemical shell[20] around the origin are generated.

Figure 1 shows a typical example of such a loopless structure. The cluster sites are shown in Fig. 1(a), and the corresponding green sites are shown in Fig. 1(b). Both pictures look very similar; the green sites appear everywhere in the cluster and their density is "proportional" to the density of the cluster.

In order to determine the critical concentration p_c, we generated an ensemble of loopless clusters and determined

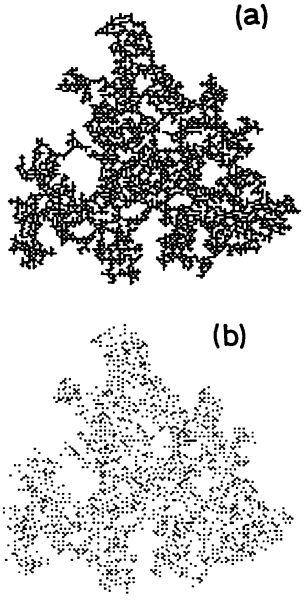

FIG. 1. A typical loopless percolation cluster at criticality: (a) the cluster sites and (b) the green sites. Note the close similarity between both pictures.

the number of "surviving" clusters of l chemical shells $N(l)$, for several values of p. In analogy to conventional percolation theory we expect that for $p > p_c$, due to the presence of an infinite cluster, $N(l)$ will reach a constant value at large l, while for $p < p_c$ it should drop exponentially. At $p = p_c$, $N(l)$ should decay with a power law,

$$N(l) \sim l^{-x}. \tag{1}$$

Figure 2 shows $N(l)$ for three representative probabilities: For $p = 0.68$ the curve approaches a constant value, while for $p = 0.64$ the curve drops exponentially. At $p = 0.66$, $N(l)$ decreases with a power law. Thus $p = 0.68$ is above p_c, $p = 0.64$ is below p_c and $p = 0.66$ is very close to p_c. More detailed simulations[22] yield $p_c = 0.660 \pm 0.002$.

In the following we consider the case $p = p_c$. For conventional percolation the number of surviving clusters of S sites scales as $N(S) \sim S^{-(\tau-2)}$, where $\tau = 1 + d/d_f$. Since the number of sites S scales with l as $S \sim l^{d_l}$, where d_l is the "chemical" dimension[20] of the cluster, it follows that

$$N(l) \sim l^{-d_l(d/d_f - 1)}. \tag{2}$$

In $d = 2$, $d_f = 91/48$ and $d_f/d_l = 1.130 \pm 0.002$,[23] yielding $d_l(d/d_f - 1) = 0.0922 \pm 0.0002$. From Fig. 2 we find that $x = 0.085 \pm 0.01$, which within the error bars is in good agreement with the accepted value for percolation. This indicates that the distribution of the loopless clusters at p_c is the same as for conventional percolation.

Figure 3(a) shows the mean number of cluster sites S and the mean number of green sites G of large loopless clusters within l chemical shells from the origin. The successive slopes shown in the inset of the figure yield $d_l = 1.68 \pm 0.02$ for both the cluster sites and the green sites. This value agrees, within the error bars, with the value $d_l = 1.678 \pm 0.003$ (Ref. 23) for conventional percolation clusters.

The mean distance R of cluster sites in the lth shell is shown in Fig. 3(b). Since $R \sim l^{d_l/d_f}$ we can deduce the ratio d_l/d_f from the figure. From the slope of the curve we obtain $d_l/d_f = 0.885 \pm 0.01$. Combining this value with our finding for d_l we obtain $d_f = 1.90 \pm 0.04$ which is very close to $d_f \cong 1.896$ of conventional percolation.

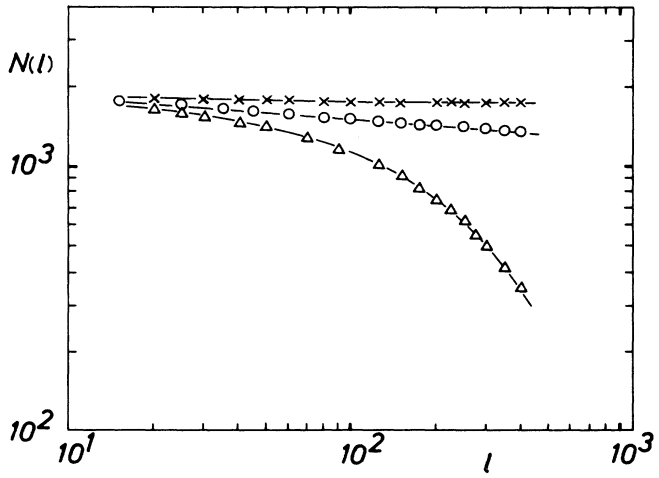

FIG. 2. Number of surviving clusters $N(l)$ containing l chemical shells, for three representative probabilities p: $p = 0.64$ (△), 0.66 (○), and 0.68 (×).

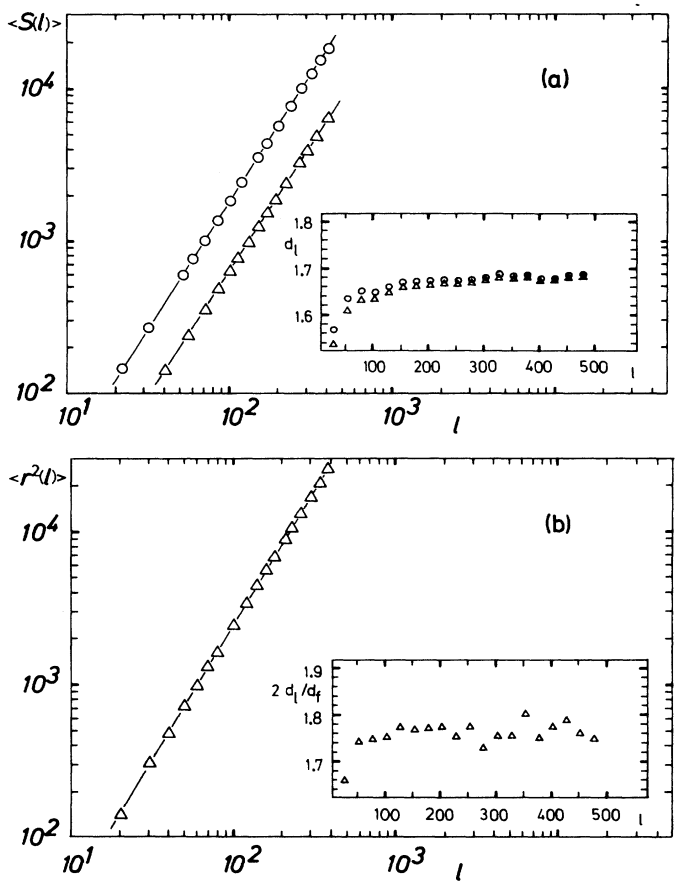

FIG. 3. (a) Plot of the mean number of cluster sites (○) and the mean number of green sites (△) at criticality as a function of the chemical distance l from the origin. The inset shows the values of the local slopes of both curves, which yield the fractal dimension d_l. (b) Plot of the mean-square radius of gyration R^2 at criticality as a function of the chemical distance l from the origin. The inset shows the values of the local slopes of the curve, which yield $2d_l/d_f$ at large l. To obtain the results, averages have been made over 10^4 configurations.

The above results indicate that the loopless percolation clusters are described by the same static exponents as conventional percolation clusters. In contrast, their transport properties are very different, since the absence of loops changes the dynamical behavior. Following Ref. 24, the resistance ρ between two sites on a loopless structure with finite ramification is simply proportional to the chemical distance l between them. Hence we have $\rho \sim l \sim R^{d_l/d_f} \sim R^{\tilde{\zeta}}$ with the resistance exponent $\tilde{\zeta}$ being simple $\tilde{\zeta} = d_f/d_l$.

Diffusion on the cluster is characterized by the mean-square displacement $\langle r^2 \rangle$ of a random walker as a function of time t, $\langle r^2 \rangle \sim t^{2/d_w}$ asymptotically. The diffusion exponent d_w is related to $\tilde{\zeta}$ by the Einstein relation $d_w = d_f + \tilde{\zeta}$.[8] The spectral dimension d_s (Ref. 8) occurring in the phonon density of states and the probability of a

random walker to return to the origin is related to d_f and d_w by $d_s = 2d_f/d_w$.

From the above considerations, using our numerical values for d_f and d_l, we find $\tilde{\zeta} = 1.13 \pm 0.01$, $d_w = 3.03 \pm 0.05$, and $d_s = 1.255 \pm 0.01$ in $d = 2$.

Anticipating that d_f and d_l are the same as for conventional percolation [where $d_f = 91/48$ and $d_l = 1.678 \pm 0.003$ (Ref. 23)], we *predict* for loopless percolation in $d = 2$

$$\tilde{\zeta} = 1.130 \pm 0.002, \quad d_w = 3.026 \pm 0.002, \quad d_s = 1.253 \pm 0.002. \tag{3}$$

These values are significantly different from $\tilde{\zeta} \cong 0.97$, $d_w \cong 2.87$, and $d_s \cong 1.33$ found for regular percolation (see, e.g., Ref. 12), because the effect of loops is to enhance transport.

In order to study the effect of loops in the whole time regime, we have performed detailed numerical simulations of random walks on the loopless clusters, using the exact enumeration method (see, e.g., Ref. 12). The result for the exponent $2/d_w$ of the mean-square displacement, as a function of time t, is shown in Fig. 4. The exponent decreases continuously with time. For times larger than $t = 1000$, $2/d_w$ is smaller than the asymptotic value for conventional percolation, reaching a value 0.672 (corresponding to $d_w = 2.98$) at about 5000 times steps. Since there is a large transient region of time in the process, it is likely that $2/d_w$ reaches its predicted asymptotic value only at very large times, being of the order of 10^5.

Above the critical concentration, we assume that the transport exponents are identical to the case $p = 1$, where the only blocked sites are the green sites. In this case loopless deterministic fractals are generated, with $d_f = d_l = d_2$ and the transport exponents can be calculated exactly, $\zeta = 1$, $d_w = d+1$, and $d_s = 2d/(d+1)$.

For all dimensions $d \geq 3$, the values for the diffusion exponent at $p = 1$ are larger than at p_c, while in $d = 1$ they trivially coincide. Anticipating the numerical value $d_f/d_l = 1.130 \pm 0.002$ (Ref. 23) in $d = 2$, we find that only in $d = 2$, d_w ($\cong 3.03$) at p_c is larger than at $p = 1$, but we do not yet have an obvious reason for this different behavior. The effect of blocking sites is to shorten branches of the cluster. Accordingly, the tendency of a random walker to get stuck in large branches of the loopless structure should be decreased at p_c, and we expect d_w at p_c not to exceed d_w at $p = 1$. This is in contrast to the prediction $d_w \cong 3.03$, which is based on the numerical result for d_f/d_l. In order to satisfy the "rule" $d_w \leq d+1$ also in $d = 2$, we must require

$$\frac{d_f}{d_l} \leq \frac{53}{48} \cong 1.104. \tag{4}$$

This result, however, is out of the error bars for d_f/d_l ($= 1.130 \pm 0.002$) found recently by Herrmann and Stanley.[23]

To conclude, we have presented a percolation model for which the static exponents are in the same universality class as regular percolation while the dynamical exponents are different from those of conventional percolation. The result indicates that from the static geometrical exponents one *cannot* obtain the dynamical exponents. An interesting question is currently studied. Do the dynamical exponents such as d_w depend *continuously* on the probability c of a green site to be blocked?

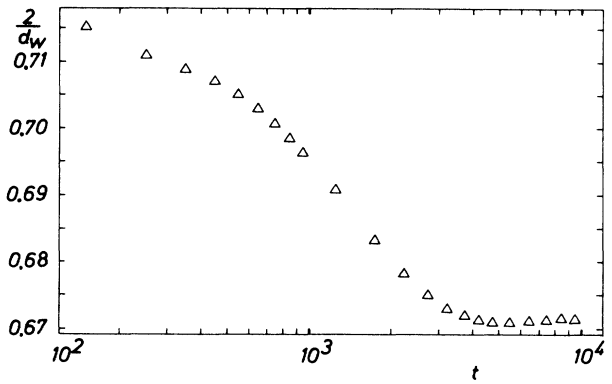

FIG. 4. The graph shows the local slope $2/d_w$ of a log-log plot of $\langle r^2(t) \rangle$ as a function of time t. To obtain this result, clusters with 250 shells have been generated and averages have been made over 400 configurations.

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