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COMMENT

Superconductivity exponent for the Sierpinski gasket in two dimensions

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Abstract. Numerical calculations based on Kirchhoff laws are used to calculate the resistance of a random mixture of conductors and superconductors on the Sierpinski gasket in two dimensions. Using modified finite-size scaling arguments we obtain for the superconductivity exponent $\tilde{s} = 0.27 \pm 0.03$, which is not predicted by any known critical exponent relations. Our method is confirmed by re-obtaining the exact known result for the conductivity exponent $\tilde{\mu}$ for the problem of a conductor-insulator mixture on the gasket.

The problem of transport on a random superconducting network (RSN) has been studied theoretically and numerically extensively in the past few years [1-13]. The conductivity Σ of such a random mixture of superconductors (with concentration p) and conductors (with concentration $1-p$) is infinite above the percolation threshold p_c. Approaching p_c from below, Σ diverges as

$$\Sigma \sim |p - p_c|^{-s} \qquad p \to p_c^- \tag{1}$$

where s is universal. This superconductivity exponent s has been of theoretical [1-5] and experimental [14-19] interest, due to its important role in transport properties of random disordered systems. The exponent s appears in the critical behaviour of the dielectric constant [14-16], in the absorption coefficient of random metal-insulator composites [17], in the conduction of binary metallic mixtures [1, 18], and in the viscosity of gels [19].

In one dimension one can show the exact result $s = 1$ [3, 5]. In two dimensions, an exact duality argument by Straley [1] shows that s is equal to the conductivity exponent μ of a mixture of conductors and insulators, defined by

$$\Sigma \sim |p - p_c|^{\mu} \qquad p \to p_c^+. \tag{2}$$

Indeed, numerical results [4] yield $s = \mu \simeq 1.3$, or $\tilde{s} = s/\nu \simeq 0.977$. In three dimensions no exact relation between s and μ is known. Numerical calculations [4] suggest that $s \approx 0.75$, or $\tilde{s} = s/\nu \simeq 0.85$, when using $\nu \approx 0.88$.

In this work we calculate the exponent s for the Sierpinski gasket in two dimensions, in order to gain some more insight about the possible relations between s and other critical exponents, either dynamic exponents such as μ, or static-geometric exponents such as ν, the exponent of percolation correlation length. Such relations were proposed in the past [2, 3, 6], but none of them seem to be valid for all dimensions [5, 6, 20].

The Sierpinski gasket is constructed from an equilateral triangle, subdividing it into four smaller triangles and taking out the central triangle. This procedure is iterated

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ad infinitum (figure 1(*a*)). The resulting Sierpinski gasket has a fractal dimension given by $2^{d_f}=3$, or $d_f=\ln 3/\ln 2=1.585$.

The attraction of the gasket is that many physical problems defined on it are exactly solvable (see, e.g., the recent review by Havlin and Ben-Avraham [21]). Though the gasket has non-trivial geometry, it has two properties, *self-similarity* and *finite ramification*, which enable an *exact* renormalisation group approach for many physical problems considered on the gasket [21-23]. The finite order of ramification of the gasket implies that the percolation threshold is $p_c=1$, which is a one-dimensional property.

The problem of a random superconducting network (RSN) on the Sierpinski gasket means that each bond is either a superconductor (probability p) or a conductor (probability $1-p$). The Sierpinski gasket was suggested as a model for the backbone of percolation cluster at criticality [22]. Thus studying the RSN problem for the gasket corresponds to a study of a mixture of conductors and insulators near criticality, where some of the conductors (concentration p) are replaced by superconductors.

A finite Siperinski gasket of order n, i.e. having length $L=2^n$ and containing 3^{n+1} bonds is constructed. The resistivity of each bond is chosen randomly, either zero resistivity with probability p (superconductor) or finite constant resistivity r with probability $1-p$ (conductor). To avoid computational difficulties, the resistivity of the superconductors was taken as 10^{-10}, which did not affect the accuracy.

In order to calculate the end-to-end resistance or conductance of the gasket, the traditional method of applying *direct* Kirchhoff laws is rather difficult, due to the complicated structure of the gasket. Therefore, one has to transform the gasket into an *effective* network, passing through triangle-star ($\Delta \rightarrow Y$) transformations. A similar method was used by Blumenfeld and Aharony [24].

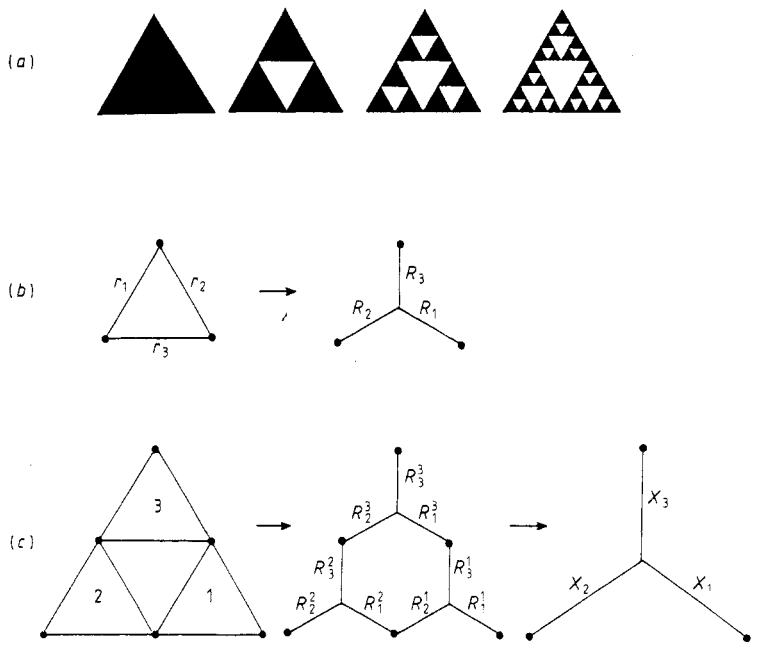

Figure 1. The Sierpinski gasket in two dimensions. (*a*) The initial triangle and first three constructions stages. (*b*) The triangle-star transformation. (*c*) The transformation to an effective network.

The first step is shown in figure 1(*b*). Here we convert each basic triangle in the gasket to a star, using the following formulae for the new resistances R_i as a function of the old ones r_i:

$$R_1 = \frac{r_2 r_3}{r_1 + r_2 + r_3}$$

$$R_2 = \frac{r_1 r_3}{r_1 + r_2 + r_3} \tag{3}$$

$$R_3 = \frac{r_1 r_2}{r_1 + r_2 + r_3}.$$

The resultant is a gasket which is entirely built of stars, instead of triangles, as shown in figure 1(*c*).

Next we replace every three connected stars (nine resistor bonds) by an equivalent star of three resistors. This is done using the following formulae (see figure 1(*c*)):

$$X_1 = R_1^1 + \frac{(R_1^3 + R_3^1)(R_2^1 + R_1^2)}{R_1^3 + R_2^1 + R_3^2 + R_3^1 + R_1^2 + R_2^3}$$

$$X_2 = R_2^2 + \frac{(R_1^2 + R_2^1)(R_3^2 + R_2^3)}{R_1^3 + R_2^1 + R_3^2 + R_3^1 + R_1^2 + R_2^3} \tag{4}$$

$$X_3 = R_3^3 + \frac{(R_1^3 + R_3^1)(R_3^2 + R_2^3)}{R_1^3 + R_2^1 + R_3^2 + R_3^1 + R_1^2 + R_2^3}$$

where superscripts denote the three original triangles.

This last procedure is repeated iteratively until the entire finite gasket is covered. As a final result we get a single star, built of three bonds (resistors) and its resistance is equivalent to that of the original gasket.

The results of such a calculation, namely the resistances x_1, x_2, x_3, were averaged over several thousands of samples. The average of these three averages is denoted by $\langle \rho \rangle$, and is used in our case as representing the resistance of the gasket.

Results for the averaged resistance $\langle \rho \rangle$ as a function of p (the concentration of superconductors) for increasing values of system size $L = 2^n$ ($n = 3, \ldots, 9$) are shown in figure 2(*a*). For $n = 3, 4, 5, 6$ the results were averaged over 30 000 samples, while for $n = 7, 8, 9$ convergence was achieved already for 5 000 samples, due to self-averaging in these larger systems.

It is interesting to point out that even for small finite systems, p_c is very close to 1, which is the precise value for the infinite gasket. Also, for $p = 0$, the RSN reduces to an ordered gasket, with all bonds having equal finite resistivity r. Our numerical results in this limit fit exactly to the scaling of resistance with length for this case, namely [21-23]:

$$\rho \sim L^{\tilde{\xi}} \qquad \tilde{\xi} = \ln(5/3)/\ln 2 \simeq 0.73\ldots \tag{5}$$

where $\tilde{\xi}$, the resistance exponent, is defined by this relation.

The above method can be easily applied to the random resistor network problem (RRN) on the gasket. For this case the bonds have random resistivity, either $r = 1$ with probability p (resistor), or infinite resistivity with probability $1 - p$ (insulator). This infinite resistivity was actually taken as 10^{10} for computational reasons, in analogy with the RSN case. After calculating the three resistances x_1, x_2, x_3, as before, we

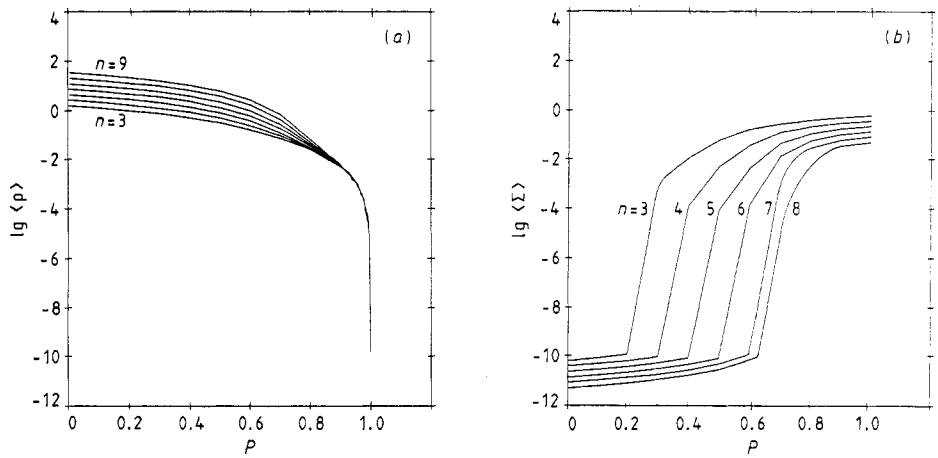

Figure 2. (*a*) RSN. Results for the averaged *resistance* (on a logarithmic scale) as a function of p, the concentration of *superconductors*), for increasing values of system size $L=2^n$ ($n=3,\ldots,9$). (*b*) RRN. Results for the averaged *conductance* (on a logarithmic scale) as a function of p, the concentration of *resistors*), for increasing values of system size $L=2^n$ ($n=3,\ldots,8$).

convert the results into three conductances Σ_1, Σ_2, Σ_3, by taking $1/x_1$, $1/x_2$, $1/x_3$, respectively, and then average those results over a few thousands of samples to get $\langle\Sigma_1\rangle$, $\langle\Sigma_2\rangle$, $\langle\Sigma_3\rangle$ and finally $\langle\Sigma\rangle$. The results are shown in figure 2(*b*).

It is worthwhile emphasising that though the RSN and the RRN are quite analogous [25], one basic difference arises immediately when comparing figure 2(*a*) with figure 2(*b*). In the RSN, p_c approaches $p_c=1$ even for very small finite gaskets, whereas in the RRN the effective percolation threshold for finite size L approaches much more slowly to the asymptotic value $p_c=1$ for infinite gaskets. The fact that in the RSN both components of the random mixture conduct (with different timescale), while in the RRN only one component conducts, has the above asymmetry effect on the percolation threshold for finite systems.

In the following we use a finite-size scaling approach similar to that presented by Stauffer [20], in order to extract the value of the critical exponent $\tilde{s}$ for the case of RSN. In general, when there are two parameters, the concentration p and the system length L, we expect every quantity varying as $|p-p_c|^x$ to follow the scaling relation

$$Q = L^{-\tilde{x}} F[(p-p_c)L^{1/\nu}]. \tag{6}$$

Equation (6) is expected to be valid in the asymptotic regime (large lengths and close to p_c), where $\tilde{x}=x/\nu$, ν is the percolation correlation length exponent, and $F(z)$ is a scaling function. The argument of this function combines the above two parameters in the form

$$z = (p-p_c)L^{1/\nu}. \tag{7}$$

The form of this combination arises from the definition of the correlation length exponent

$$L \sim (p-p_c)^{-\nu} \tag{8}$$

which implies

$$(p-p_c)L^{1/\nu} \approx \text{constant}. \tag{9}$$

This definition combines p and L through the exponent ν and is valid for most properties of percolation.

In the present case, percolation on the Sierpinski gasket in 2D, an *exact* relation between p and L, *different* from the general one, was found by Gefen *et al* [23]. Using an exact renormalisation group procedure, they found

$$L \sim \exp\left(\frac{\frac{1}{4}\ln 2}{(p_c - p)^2}\right) \tag{10}$$

yielding

$$(p_c - p)\sqrt{\ln L} \approx \text{constant}. \tag{11}$$

Therefore, a modification of the scaling variable in equations (6)-(7) is required. The difference between equations (9) and (11) implies that one has to replace $L^{1/\nu}$ by $\sqrt{\ln L}$. Thus the final form of the general scaling relation for the Sierpinski gasket becomes

$$Q = L^{-\tilde{s}} F[(p_c - p)\sqrt{\ln L}]. \tag{12}$$

where $p_c = 1$.

To test the validity of this modified finite-size scaling, we apply it first to the RRN problem. For this case, the conductance is defined by (2), and (12) is

$$\Sigma = L^{-\tilde{\mu}} F[(1-p)\sqrt{\ln L}]. \tag{13}$$

The exponent $\tilde{\mu}$ was calculated by Gefen *et al* [22], and its value is $\tilde{\mu} = \ln(5/3)/\ln 2 = 0.73$. The idea of the finite-size scaling approach, expressed by (13), is that if we plot $\Sigma L^{+\tilde{\mu}}$ as a function of $(1-p)\sqrt{\ln L}$, with the right value of $\tilde{\mu}$, the results for different sizes L should be represented by a single curve $F(z)$, in the region $p \to p_c = 1$.

Indeed, figure 3 verifies that this happens for $\tilde{\mu} = 0.73$ and close to $p_c = 1$. For other close values of $\tilde{\mu}$ the curves do not coincide.

Next we apply this method to the RSN case. In this case the resistance is defined through (1), and by analogy we expect

$$\rho \sim L^{-\tilde{s}} G[(1-p)\sqrt{\ln L}]. \tag{14}$$

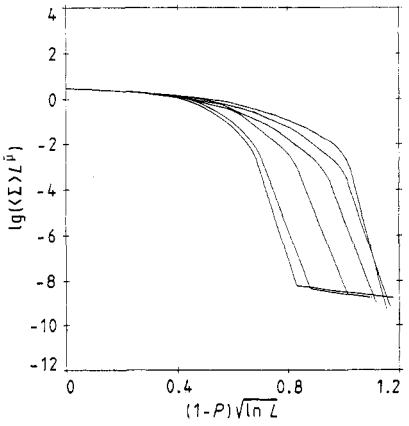

Figure 3. RRN. Scaling of $\langle\Sigma\rangle L^{\tilde{\mu}}$ (on a logarithmic scale) as a function of $(1-p)\sqrt{\ln L}$. A single curve in the region $p \to 1$ is obtained with the exact known value $\tilde{\mu} = 0.73\ldots$.

Plotting $\rho L^{+\tilde{s}}$ as a function of $(1-p)\sqrt{\ln L}$ (figure 4), the best scaling for $p \to 1$ is achieved with the value

$$\tilde{s} = 0.27 \pm 0.03. \tag{15}$$

This value is not predicted by any of the suggestions for critical exponents relations [2, 3, 5, 6, 20].

Combining the above result for $\tilde{s}$, and the well known result for $\tilde{\mu}$, we get, within the statistical error,

$$\tilde{s} + \tilde{\mu} \simeq 1. \tag{16}$$

The combination $s+\mu$ is discussed by Straley [2]. He proposed the relation

$$s + \mu = d\nu \tag{17}$$

and argued that both s and μ should be given equally important placement in an exponent relationship, due to their equal footing in the theory. For example, this combination plays an important role in the theory of inhomogeneous conductors similar to that of the gap exponent in the theory of critical phenomena. It appears also in some of the recent theoretical works on the RSN–RRN problems [3, 8, 11, 13].

Relation (16) is known to be *exact* for one dimension [2, 13], where it agrees with (17). The common feature of one-dimensional systems and the Sierpinski gasket is their percolation threshold $p_c = 1$, due to their finite order of ramification. Therefore, the gasket is regarded in many aspects as a quasi-one-dimensional case [13, 23]. Hence we suggest that relation (16) may be another common property of finitely ramified systems. This can be supported by the theoretical approach to the RSN in one dimension by Leyvraz *et al* [13], which shows the peculiar properties for systems with $p_c = 1$.

Another interesting question is the form of the scaling function $G(z)$ in (14). Considering first the RRN case, the conductivity Σ decreases when system size L increases. This is demonstrated by the factor $L^{-\tilde{\mu}}$ in the scaling relation (13), which is dominant upon the scaling function $F(z)$. This function becomes a finite constant at $z = 0$ ($p = p_c$), and vanishes for $z \to \infty$ ($L \to \infty$) [20].

But this is not the case for the RSN problem. Our numerical results show that the resistance *increases* with system size L, while the factor $L^{-\tilde{s}}$ in (14) seems to argue the

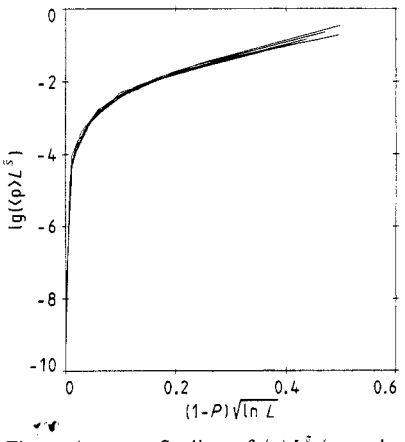

Figure 4. RSN. Scaling of $\langle\rho\rangle L^{\tilde{s}}$ (on a logarithmic scale) as a function of $(1-p)\sqrt{\ln L}$. The best scaling (near $p = 1$) is obtained with $\tilde{s} = 0.27$ (± 0.03).

opposite. Hence it is suggested that in this case the scaling function $G(z)$ is the dominant factor in the scaling relation. For $z = 0$ ($p = p_c$) it vanishes, but for p below p_c, its strong effect causes the resultant increase of the resistance. This scaling function might have an exponential dependence on some power of z, and its exact form is now under research. Anyway, the fact that the scaling function plays here the main role, and not $L^{-\tilde{s}}$, might be one of the reasons for the difficulties in deriving an exact analytical expression for the superconductivity exponent for the gasket.

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