COMMENT

Percolation thresholds on finitely ramified fractals

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Received 7 August 1989, in final form 28 September 1989

Abstract. Exact renormalisation group recursion relations are used to estimate the effective percolation thresholds for site and bond percolation on finite-generation Sierpinski gaskets, and for bond percolation on branching Koch curves.

The Sierpinski gasket (SG) is a prototype of a finitely ramified fractal [1] which often served as a theoretical 'laboratory' for concepts related to fractals. In particular, Gefen et al [1] were the first to treat percolation on a SG, using an approximate renormalisation group (RG) recursion relation. They found that \( p_c = 1 \), a result which is intuitively plausible given the low connectedness of the SG. More precisely, let us look at finite-generation approximations of a SG, and let us call \( R_n \) the probability that on an \( n \)th generation \( \text{SG} \) all corners are connected. We then define an effective threshold \( p_c^{(n)} \) by requiring \( R_n(p = p_c^{(n)}) = c \), with \( 0 < c < 1 \). In [1] it was found for bond percolation that

\[
p_c^{(n)} \approx 1 - 1/2 \sqrt{n} \quad \text{for } n \to \infty. \tag{1}
\]

The site percolation problem has been studied more recently by Yu and Yao [2], who found \( p_c^{(n)} \approx 1 - 1/n \) by means of heuristic arguments and numerical simulations. Related to these problems are other transport problems on the SG, treated in [3-6].

It is the purpose of this comment to point out that for percolation on a SG one can give the exact RG recursion relations, similar to those given in [3] for the problem of Joule heat distribution on a SG, and in [5, 6] for self-avoiding walks and trails.

In addition to the probability \( R_n \) for percolation from any corner to both others, we need the probability for percolation between two corners, but not between them and the third. We call this \( S_n \). Obviously, \( 1 - R_n - 3S_n \) is the probability that there is no percolation between any pair of corners. Graphically, we represent \( R_n \) and \( S_n \) as shown in figure 1.

![Figure 1. Probabilities for a finite-generation Sierpinski gasket to percolate: (a) from any corner to any other corners; (b) from corner A to corner B, but not to corner C.](image-url)
For bond percolation, the RG recursion for $R_n$ is shown graphically in figure 2. Together with the somewhat more complicated recursion for $S_n$, we then obtain the exact relations

$$R_{n+1} = R_n^3 + 6R_n^2S_n + 3R_nS_n^2$$
$$S_{n+1} = (R_n + S_n)^2 - 4R_n^2S_n + S_n^3 - R_n^3.$$  \(2\)

We make now an ansatz

$$R_n = 1 + \alpha/n + O(n^{-3})$$
$$S_n = \beta/n + \gamma/n^2 + O(n^{-3})$$  \(3\)

with open parameters $\alpha$, $\beta$ and $\gamma$. Notice that no term $\sim 1/n^2$ appears in the ansatz for $R_n$, as such a term can always be absorbed in the term $\sim 1/n$ by a translation $n \to n + \text{constant}$. The recursion relations give the unique solution

$$\alpha = -\frac{1}{4}, \quad \beta = -\frac{1}{4}, \quad \gamma = -\frac{1}{16}.$$  \(4\)

In order to have non-negative probabilities, we can use this solution only for $n < 0$. Level $n = 0$ corresponds to the outer length scale. Assume now that the recursions (2) hold only for $n > -N$, i.e. level $n = -N$ corresponds to the inner length scale. At this scale, we have a simple triangle with bond probability $p$, i.e.

$$R_{-N} = p^3 + 3p^2(1-p)$$
$$S_{-N} = p(1-p)^2.$$  \(5\)

Comparing (3) and (5) gives then in agreement with [1]

$$p_c^{(N)} = 1 - 1 - \sqrt{N} + O(N^{-1})$$ \hspace{1cm} \text{(bond percolation).} \hspace{1cm} \(6\)

This result is supported by numerical simulations which were performed using a technique described in detail in [4]. The effective percolation threshold was determined according to the condition $R_n(p = p_c^{(n)}) = 0.95$, where the constant $c = 0.95$ was chosen arbitrarily.

Figure 2. Recursion relation for $R_n$, the probability to percolate from any corner to any other.

Figure 3. Recursion relation defining a branching Koch curve.
For site percolation, the recursion relations are somewhat more complicated. A straightforward analysis gives
\[
R_{n+1} = R_n^3 p^3 + 3R_n p^2 ((1-p)R_n^2 + 2R_n S_n + S_n^2)
\]
\[
S_{n+1} = p[(S_n + R_n)^2 + pS_n^3 - p(3+p)R_n^2 S_n - p(2-p)R_n^3].
\]
(7)

We were not able to solve this analytically as in the bond percolation case. It is however trivial to iterate (7) numerically, with the initial values for \( R \) and \( S \) given by (5). From such iterations, we found
\[
p_c^{(N)} \approx 1 - 0.5/N \quad \text{site percolation}
\]
(8)
which agrees qualitatively but not quantitatively with the result of [2]. We might add that we also performed numerical iterations on (2), thereby verifying (3)-(6).

Finally, we should mention that similar (and indeed simpler) exact recursion relations can be given for many other fractals, including in particular branching Koch curves [7]. In the latter case, one finds in general an exponential convergence of \( p_c \) towards 1. For instance, for bond percolation on the branching Koch curve shown in figure 3 we get a RG relation for the probability \( R_n \) of percolation
\[
R_{n+1} = R_n^3 (1-R_n)
\]
(9)
from which we obtain \( p_c^{(N)} \approx 1 - \text{constant}/2^N \). Again, this result is found to be in perfect agreement with numerical simulations.

References