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COMMENT

Distribution of first-passage times for diffusion at the percolation threshold

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Abstract. Simulations for dispersion of diffusion at the percolation threshold of triangular and Bethe lattices show scaling behaviour. With 'topological' bias we find a maximum of the arrival time distribution at short times, a power-law decay for intermediate times and an exponential decay for long times.

If fluids flow through a porous medium, different parts of the fluid take different amounts of time to flow the same distance (dispersion). One model for dispersion is diffusion on percolating clusters [1-5], where a random walker can move only on occupied sites. This walk is called biased if one direction is taken more often than the others. This direction can be fixed in space [6], oriented away from the origin ('topological') [7], oriented along the current flow direction [8, 9], or it can be random [10]. The case of topological bias seems numerically and analytically best understood [7] and thus is chosen for the present study.

Therefore we check how long a random walker needs to travel a 'chemical distance' l, i.e. to move to a site separated by l nearest-neighbour bonds (within the percolating cluster) from the origin of the walk. $P(t)$ is the probability that the walker arrives there first after t steps. In general, a step which increases the chemical distance l from the origin is taken with a probability proportional to $1+E$, a step in the opposite direction with probability proportional to $1-E$. This bias field may correspond to the pressure gradient in a porous medium, if a fluid is injected at the origin. We simulate this dispersion problem on a computer at the critical concentration $p=p_c=\frac{1}{2}$ of a triangular and a Bethe lattice (Cayley tree). The random medium is produced by Monte Carlo methods, the diffusion process on it by exact enumeration [2].

Figure 1 shows that the histogram $P(t)$ of first-arrival times obeys a scaling law even for moderately large distances l. The RMS fluctuation $(\langle t^2\rangle-\langle t\rangle^2)^{1/2}$ is about as large as the average $\langle t\rangle$. We plot double logarithmically the ratio $\pi(t)=P(t)/P(t_{\max})$ against $t/t_{1/2}$. Here $t_{\max}$ is the time at which $P(t)$ reaches its maximum, and $t_{1/2}$ the later time after which $P(t)$ has decayed to half its maximum value. This way of plotting avoids any assumptions on how the times depend on the length l. The inserts in figure 1 show that $t_{\max}$ and $t_{1/2}$ increase roughly as $l^{2.4}$ on the triangular lattice and as $l^{2.6}$ on the Cayley tree. Theoretically we expect [2] these exponents to be about $d_w^l=2.5$ and $d_w^l=3$ for $t\to\infty$.

We see an impressive agreement between the triangular and Bethe lattices. For example, the ratio $t_{1/2}/t_{\max}$ is about 3 in the triangular lattice and only 10% larger in

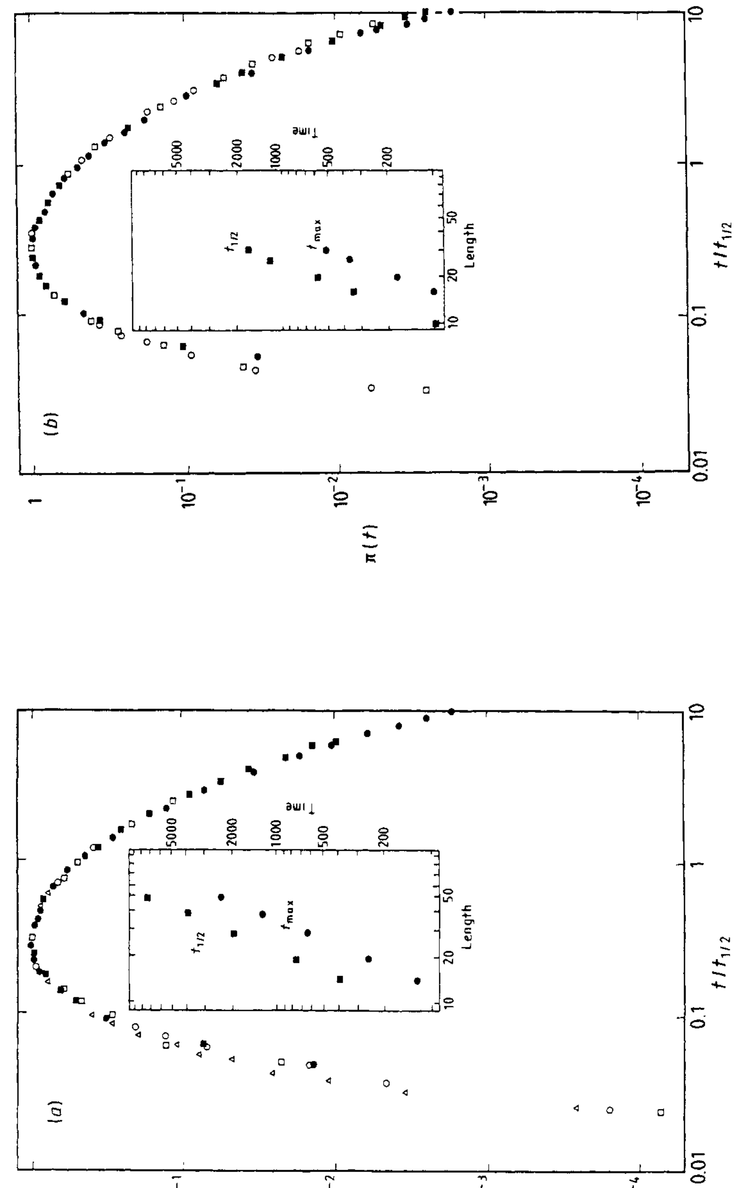
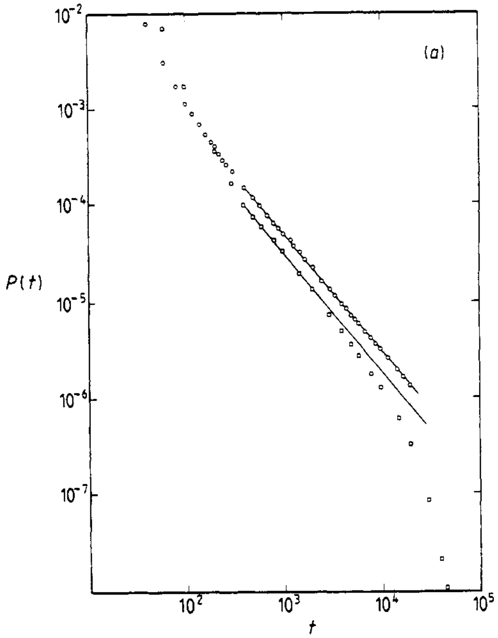


Figure 1. Scaled histogram $\pi(t) = P(t)/P(t_{max})$ of arrival times against scaled time $t/t_{1/2}$ for various chemical distances l. The insert shows the variation of characteristic times with chemical length l. (a) Refers to the triangular lattice: $l = 15$ (●), 20 (■), 30 (□), 40 (○) and 50 (△); (b) refers to the Cayley tree: $l = 16$ (●), 20 (■), 26 (□), 30 (○).



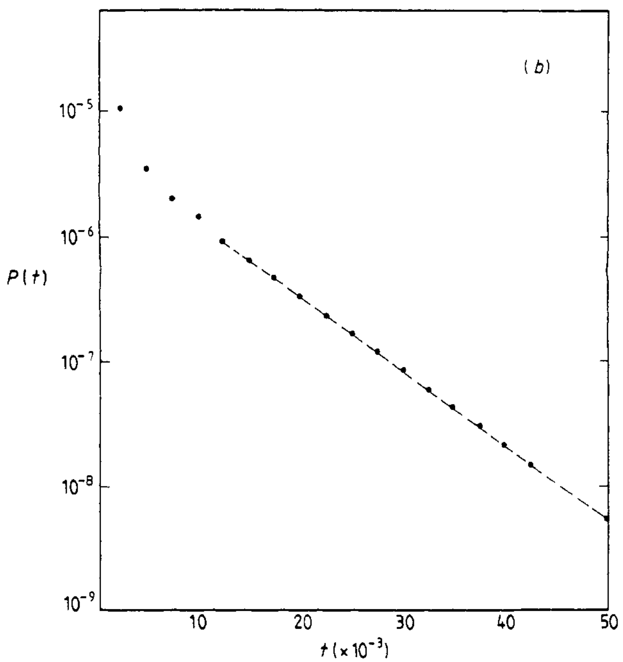


Figure 2. (a) Histogram $P(t)$ for the triangular lattice for $l = 35$, $E = 0.8$(○), and for $l = 10$, $E = 0.8$(□). In both cases a power-law regime of $P(t) \sim t^{-1.2}$ is seen. In the case $l = 10$ the exponential decay for $t > 10^4$ is seen clearly in (b) where $\ln P(t)$ is plotted against t.

the Bethe lattice. In both cases the data for different l fall into the same curve except for very small $\pi(t)$. Roughly, this curve is a parabola, corresponding to a log-normal distribution of arrival times:

$$\log P(t) \propto [\log(t_{max}) - \log(t)]^2. \tag{1}$$

However, a slight asymmetry is visible, and the log-normal distribution should not be expected to be asymptotically exact. For example, if $t \to \infty$ at fixed l we expect [11] $P(t)$ to decay exponentially, as confirmed by data on $l = 10$ (Cayley tree) for $\pi(t) < 10^{-6}$ (not shown). The first-passage-time distribution $P(t)$ can be related to the distribution of voltage drops between the site at the origin of the walker and a site at chemical distance l. Since for the voltage-drop problem an infinite hierarchy of exponents are needed to characterise the different moments, it is expected that for this case an analogous hierarchy of exponents will characterise the moments $\langle t^n \rangle$.

With a non-zero bias E the results become more complicated. The most probable time t_{max} of arrival shifts, for strong fields ($E \to 1$), towards l, which is the minimum time to traverse l bonds. For t somewhat larger than t_{max}, the arrival probability $P(t)$ falls rapidly. If l is large enough (e.g., $l = 35$ but not $l = 10$) we then see a regime where $P(t)$ decays less strongly, roughly like $1/t$. Finally, for $t \to \infty$ exponential decay is expected, and is seen explicitly in our longest computer run. Figure 2 summarises some of our data. The intermediate regime with its power-law behaviour can be explained as follows. It has been shown [12] that for a walker having a waiting time distribution $\phi(t) \sim t^{-\alpha}$ in a finite system surrounded with traps, the first-passage-time probability $P(t)$ also scales as $t^{-\alpha}$. This is analogous to our case. To calculate α we make use of a recent result [13] found for topological biased diffusion on percolation:

$$P_0(w) \sim \frac{1}{w(\ln w)^{1+\gamma}}. \tag{2}$$

Here $P_0(w)$ is the distribution of transition rates w to pass a dangling end along the backbone of the cluster due to the delays made by visiting in the dangling ends. From (2), and since $w \sim t^{-1}$, we find

$$\phi(t) \sim \frac{1}{t(\ln t)^{1+\gamma}}. \tag{3}$$

This result predicts $P(t)$ to be proportional to $1/t$ with logarithmic corrections. Indeed, the power calculated from figure 2 is $P(t) \sim t^{-1.2}$ which may indicate the effect of logarithmic corrections. The crossover to exponential decay for $t \to \infty$ is also understood: since the system is finite there is a minimum cutoff for equation (2), w_{min}, and, for $t \gg w_{min}^{-1}$, $P(t)$ should decay exponentially. The power-law regime might correspond to $1/f$ noise if Fourier transforms of the current fluctuations are observed [10, 11]. It would be interesting to search for similar effects in other types of bias [14, 15].

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References

[1] de Gennes P G 1976 *La Recherche* 7 919
Kopelman R 1976 *Topics in Applied Physics* vol 15, ed F K Fong (Berlin: Springer)
Mitescu C and Roussenq J 1983 *Percolation Structures and Processes* ed G Deutscher, R Zallen and J Adler (Bristol: Adam Hilger) p 81

[2] Havlin S and Ben-Avraham D 1987 *Adv. Phys.* to be published
[3] Haus J W and Kehr K W 1987 *Phys. Rep.* **150** 263
[4] Gefen Y, Aharony A and Alexander S 1983 *Phys. Rev. Lett.* **50** 77
[5] Ben-Avraham D and Havlin S 1982 *J. Phys. A: Math. Gen.* **15** L691
[6] Böttger H and Bryskin V V 1982 *Phys. Status Solidi* B **113** 9
Barma M and Dhar D 1983 *J. Phys. C: Solid State Phys.* **16** 1451
Ohtsuki T 1982 *J. Phys. Soc. Japan* **51** 1493
[7] Havlin S, Bunde A, Stanley H E and Movshovitz D 1986 *J. Phys. A: Math. Gen.* **19** L693
[8] Roux S, Mitescu C D, Charlaix E and Baudet C 1986 *J. Phys. A: Math. Gen.* **19** L687
Roussenq J, Mitescu C and Stauffer D 1986 *Preprint*
[9] de Arcangelis L, Koplik J, Redner S and Wilkinson D 1986 *Phys. Rev. Lett.* **57** 996
[10] Pandey R B 1987 *J. Phys. A: Math. Gen.* **20** L337
[11] Goldhirsch I and Noskovitz S H 1987 *J. Stat. Phys.* to be published
[12] Weiss G H and Havlin S 1987 *Phil. Mag.* to be published
[13] Havlin S, Bunde A, Glaser Y and Stanley H E 1986 *Phys. Rev.* B **34** 3492
[14] Michel G 1986 *J. Phys. A: Math. Gen.* **19** 2461
[15] Sahimi M, Davis H T and Scriven L E 1983 *Chem. Eng. Commun.* **23** 329