Stable and Metastable States in Congested Traffic

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Abstract. A new single lane inertial car following model of traffic flow is presented. The model demonstrates the presence of three phases in traffic flow: free flow, non-homogeneous congested flow and homogeneous congested flow. In the nonhomogeneous congested flow we find many periodic stable states with different values of flux and wavelength. We also find that states with relatively low and relatively high values of wavelength are metastable.

1 Introduction

Traffic dynamics has traditionally been modelled using either fluid dynamics approach or microscopic car-following rules [1–6]. In the present paper we investigate a new inertial car-following model recently introduced [7]. The results are in good agreement with experimental findings recently reported by Kerner and Rehborn [8–10].

We assume that the car acceleration is affected by four factors:

(a) aspiration to keep safety time gap $T$ from the car ahead,
(b) pre-braking if the car ahead is much slower,
(c) aspiration not to exceed significantly the permitted velocity $v_{per}$ (speed limit),
(d) random noise $\eta$.

The acceleration of the $n$th car $a_n$ is therefore given by a sum of four terms depending on its coordinate $x_n$, velocity $v_n$, distance to the car ahead $\Delta x_n = x_{n+1} - x_n$ and the velocities difference $\Delta v_n = v_{n+1} - v_n$:

$$a_n = A \left( 1 - \frac{\Delta x_n^0}{\Delta x_n} \right) - \frac{Z^2(-\Delta v_n)}{2(\Delta x_n - D)} - kZ(v_n - v_{per}) + \eta, \quad (1)$$

where $A$ is a sensitivity parameter, $D$ is the minimal distance between consecutive cars, $k$ is a constant, $\Delta x_n^0 = v_n T + D$ is the safety distance. The function $Z$ is defined as $Z(x) = (x + |x|)/2$. In further analytical and numerical exploration of the model the noise term $\eta$ is omitted unless otherwise stated.
In the following we discuss in more detail the terms in the right side of (1).

The first term plays an important role when the velocity difference between consecutive cars is relatively small. In this case the n-th car accelerates if \( \Delta x_n > \Delta x_n^0 \) and brakes if \( \Delta x_n < \Delta x_n^0 \).

The second term is essential when \( v_n \gg v_{n+1} \). According to the first term a car getting close to a much slower car brakes only if \( \Delta x_n < \Delta x_n^0 \). This term enables it to start braking even at a bigger distance.

The dissipative third term represents a repulsive force acting when the velocity exceeds the permitted velocity.

In the deterministic case the motion of cars is described by the following system of ordinary differential equations

\[
\begin{aligned}
x_n &= v_n, \\
v_n &= A \left( 1 - \frac{v_n - T \frac{x_{n+1}}{x_{n+1}}}{x_{n+1}} \right) - \frac{Z^2(v_n - v_{n+1})}{2(x_{n+1} - x_n - D)} - k \frac{Z(v_n - v_{per})}{v_n^0},
\end{aligned}
\]

(2)

\( n = 1, \ldots, N \) with periodic boundary conditions \( x_{N+1} = x_1 + \frac{N}{\rho}, v_{N+1} = v_1 \).

A solution of Eqs. (2) which corresponds to the homogeneous flow is

\[
v_n^0 = v_0^0 = \begin{dcases}
\frac{A(1-D_0 \rho + k v_{per})}{\rho T + k}, & \rho \leq \frac{1}{D + T v_{per}};

\frac{1-D_0}{\rho T}, & \rho \geq \frac{1}{D + T v_{per}}.
\end{dcases}
\]

(3)

In the following numerical analysis we use parallel updating rule and parameters values \( A = 3 m/s^2, v_{per} = 25 m/s, T = 2 s, D = 5 m \), and \( k = 2 s^{-1} \). For the dimensionless model we have only three parameters: \( A, v_{per}, \) and \( k \).

2 Three phases in traffic flow

Results of the deterministic model simulations are presented in Figs. 1(a,b). The data plotted on these figures correspond to the state of the system after a transient time, starting with random and (nearly) homogeneous initial conditions. The flux-density relation plotted in Fig. 1(a) (the fundamental diagram) consists of two curves: the first, increasing part starting at the origin, which represents free flow and the second part which corresponds to congested flow. Fig. 1(b) illustrates the dependence on density of mean square variation of velocities.

\[
\sigma_v = \left[ \frac{1}{N} \sum_{n=1}^{N} (v_n - \langle v \rangle)^2 \right]^{1/2},
\]
Fig. 1. Three phases of traffic flow, and bistability regime. (a) Fundamental diagram and (b) Mean square variation of velocities for homogeneous initial conditions (black circles) and nonhomogeneous initial conditions (white diamonds). Parameters values are $A = 3 \text{ m/s}^2$, $v_{\text{max}} = 25 \text{ m/s}$, $k = 2 \text{ s}^{-1}$, $T = 2 \text{ s}$, $D = 5 \text{ m}$, and $N = 37$. (c) Qualitative plot of $S(\rho)$.

divided by the average velocity $\langle v \rangle$. From this figure it can be seen that congested traffic consists of two different phases, homogeneous and nonhomogeneous.

Therefore we can derive the existence of three phases in traffic flow which is in good agreement with experimental observations [9,10] and with results of other models. The three phases are:

1. free flow (for $\rho < \rho'$),
2. nonhomogeneous congested flow (for $\rho_1 < \rho < \rho_2$),
3. homogeneous congested flow (for $\rho > \rho_2$).

In the first and the third regimes $\sigma_v = 0$ which means that the flow is homogeneous. In the second regime the flow is nonhomogeneous (even in absence of noise in drivers behavior). The characteristics of nonhomogeneous congested flow are discussed in details in Section 4. In the range of densities $\rho_1 < \rho < \rho'$ both homogeneous and nonhomogeneous states are stable, which is also in agreement with experimental results [8] and with results of other models (e.g. [3–5,11]).

3 Stability of homogeneous flow

In this section we give a stability analysis of the homogeneous flow solution (3) of equations (2). The linearization of Eqs. (2) near the homogeneous flow solution (3) in variables $\xi_n = x_n - x_n^0$ has the form

$$\xi_n = -p\xi_n + q(\xi_{n+1} - \xi_n), \quad n = 1, \ldots, N,$$

where $\xi_{N+1} = \xi_1$, $p = AT\rho + k$, $q = \frac{AT + kT\rho + kD}{AT\rho + k}$ $\cdot Ap^2$ for $\rho \leq \frac{1}{AT\rho + k}$ and $p = AT\rho$, $q = Ap$ otherwise.
As in [3], a solution of equation (4) can be written as

\[ \xi_n = \exp\{i\alpha n + zt\}, \]  

(5)

where \( \alpha = \frac{2\pi}{N\kappa} (\kappa = 0, \ldots, N - 1) \) and \( z \) - a complex number. Substituting (5) into (4) we obtain the algebraic equation for \( z \)

\[ z^2 + p z - q(e^{i\alpha} - 1) = 0. \]  

(6)

Each of the \( N \) equations (6) has two solutions. These \( 2N \) different complex numbers are the eigenvalues of system (4). One of them (which corresponds to \( \kappa = 0 \)) is equal to zero regardless of values of parameters. In this case all \( \xi_n \) in (5) are equal to a constant and belong to the one-dimensional subspace of equilibria of system (4) (defined by equations \( \xi_1 = \ldots = \xi_N \), \( \xi_1 = \ldots = \xi_N = 0 \)). This indicates that the disturbed state \( x_n \) for \( z = 0 \) is also homogeneous. For \( z \neq 0 \), \( \xi_n \) in (5) is a wave with increasing or decreasing amplitude. Therefore, if we find conditions under which other \( 2N - 1 \) eigenvalues have negative real parts (the magnitude of wave (5) decreases with time) we can say that under these conditions the homogeneous flow solution (3) is stable.

Following the approach of [3] we can derive this condition as \( \frac{2}{\kappa^2} > 2 \) or

\[ S(\rho) > 2, \]  

(7)

where

\[ S(\rho) = \begin{cases} \frac{(ATp + k)^2}{\rho^2 A (AT + kW_{cr} + BD)}, & \rho < \frac{1}{AT + kW_{cr}} \\ A\rho^2, & \rho > \frac{1}{AT + kW_{cr}} \end{cases} \]

A qualitative plot of \( S(\rho) \) is sketched in Fig.1(c). Condition (7) implies that the homogeneous flow state is stable for \( \rho' < \rho < \rho'' \), where \( \rho' = \frac{1}{AT + kW_{cr}} \) and \( \rho'' = \frac{2}{\kappa^2} \). Simulations show that \( \rho'' \approx \rho_2 \).

4 Nonhomogeneous states and their stability

Our simulations show that for every given value of density in the nonhomogeneous congested flow regime there exist many stable periodic states with different wavelengths. These states correspond to different limit cycles of system (2). Figs. 2(a-c) present three of these states for the same value of density. Shown are the cars velocities after the nonhomogeneous flow regime has stabilized for three different initial conditions. The dependence of the flux on the wavelength \( \lambda_N \) is shown in Fig. 2(d). Here \( \lambda_N \) is the wavelength in units of number of cars.

The last finding, namely the existence of a range of possible flux values for every given density, implies that nonhomogeneous congested flow displays a two-dimensional region in the flux-density plane, as was found experimentally
by B. S. Kerner [10]. This two-dimensional region is shown in Fig. 2(e). This figure, which is qualitatively similar to corresponding figures in [10], was obtained by simulations with different harmonic initial conditions. But we also find that the two-dimensional region in the fundamental diagram consists of many curves and each of them corresponds to a different wavelength \( \lambda_N \). Some of these curves are shown in Fig 2(f).

Simulations of the model with nonzero white noise term \( \eta \) in Eq. (1) show that different states have different sensitivity to noise. We define the noise threshold \( \eta_{th} \), above which the average wavelength is not preserved. From Fig. 3(a) it can be seen that \( \eta_{th} \) is higher for the states with intermediate wavelengths than that for states with relatively high and relatively low values of wavelength. Applying noise with amplitude slightly above this threshold can shift the system from a periodic state to a more stable state which is less sensitive to noise. The space-time diagram on Fig. 3(b) illustrates this transition. After the noise term was applied at \( t = 200s \) the system moved from a metastable state with \( \lambda_N = 5 \) to a more stable state with \( \lambda_N = 8 \). The latter state is closer to the most stable state, which is denoted with an empty circle in Figs. 3(a) and 2(d).
Fig. 3. (a) Noise sensitivity of stable states ($\rho = 0.06 \text{veh/m}$). The empty circle denotes the most stable state. (b) Transition from a metastable to a stable cycle in presence of noise. The global density is $\rho = 0.06 \text{veh/m}$, noise is added at $t = 200s$.

References