PHYSICAL REVIEW B VOLUME 37, NUMBER 10 1 APRIL 1988

Dynamics in multiplicative processes

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(Received 9 November 1987)

We study the dynamics on fractals generated by multiplicative processes. The fractal's elements are the transition rates $\{w_i\}$ in one-dimensional systems. We find a general relation for dynamics $d_w = 1 - \tau(1)$, where d_w is the exponent characterizing the mean-square displacement, $\langle x^2 \rangle \sim t^{2/d_w}$, and $\tau(q)$ characterize the scaling of the qth moment of the fractal elements with its size. We show that criticality in d_w occurs only when $f(\alpha)$ spectra are discrete.

One of the basic properties of the multiplicative process is its description by a set of generalized dimensions.[1] An alternative description[2] is the spectrum of scaling exponents of the fractal $f(\alpha)$. These properties are found[3-7] in a variety of areas (e.g., chaotic dynamical systems, fractal growth processes, percolation, turbulence, etc.). A recent suggestion[8] was made to give a dynamical meaning to a fractal by mapping the process of refinement of the fractal measure to transfer matrix theory of a certain Ising-spin model. All attempts to connect the fractal exponents spectrum to the dynamical systems have been carried out for deterministic systems. In this Rapid Communication, we present a relation between the static properties of the fractal and stochastic motion. We show that a discrete $f(\alpha)$ spectrum represents a dynamical transition in the diffusion exponent d_w defined by $\langle x^2 \rangle \sim t^{2/d_w}$ where $\langle x^2 \rangle$ is the mean-square displacement of a random walker. On the other hand, a continuous $f(\alpha)$ spectrum does not possess criticality in the dynamics. In one-dimensional (1D) systems, the diffusion exponent d_w can be related[9] to the autocorrelation function $P_0(t)$ by $P_0(t) \sim t^{-2/d_w}$, and to the conductivity $\sigma(\omega) = (n_0 e^2/k_B T)\langle D(-i\omega)\rangle$, where the mobility[10] is given by

$$D(\omega) = (\omega^2/2)\int_0^\infty x^2 e^{-\omega t} dt \ .$$

These relations demonstrate the importance of criticality of d_w in physical properties. The main idea of the present work lies in treating the transition probabilities of the particle as fractal objects. In particular, we examine two kinds of systems: hierarchical and multifractal.

Consider the partition function

$$\Gamma(\tau,q) = \sum_{i=1}^{N} \frac{p_i^q}{l_i^q} \ , \tag{1}$$

where N is the number of segments with length l_i at the nth iteration in the fractal construction process and p_i is the measure of the segment. It has been shown that if $n \to \infty$, i.e., $\max(l_i) \to 0$ then $\Gamma(\tau,q)$ is finite only if

$$\tau(q) = q\alpha - f \equiv D_q(q-1) \ , \tag{2}$$

where α and f describe the singularity and the effective dimension of the qth moment of the measure. In the self-similar processes discussed here, we take all l_i to be equal and their total number must be the number of segments N of iteration n, thus, $l_i = l = N^{-1}$. We define the measure of a segment p_i to be inversely proportional to $W_{i,i\pm 1}$ where $W_{i,i\pm 1}$ is the transition rate for a particle to hop from site i to its nearest neighbors. Note that while previous studies assumed that $\sum_i p_i$ is normalized for all iterations, in the present study $\sum_i p_i$ is not necessarily normalized. It can be shown that Eq. (2) is valid for the general case $\sum_i p_i \geq 1$. While for $\sum_i p_i = 1$ it was shown by Hentschel and Procaccia[1] that D_1 is finite and $\tau(1) = 0$, it can be shown that when $\sum_i p_i > 1$, D_1 diverges and $\tau(1) \neq 0$ where $\tau(1)$ represents the "fractal dimension" of the transition rates.

What information can one obtain from $\tau(q)$ and $f(\alpha)$ spectra regarding the physical properties of the system? Using the relation,[11] valid in the limit $t \to \infty$,

$$\frac{1}{D} = \frac{t}{\langle x^2 \rangle} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{W_i} \ , \tag{3}$$

where N is the number of distinct sites visited by the walker, we find

$$t \sim \langle x^2 \rangle N^{-[1+\tau(1)]} \ . \tag{4}$$

Identifying (in 1D) $N \sim \langle x^2 \rangle^{1/2}$, we find, asymptotically ($t \to \infty$),

$$d_w = 1 - \tau(1) \ . \tag{5}$$

Therefore, the question of the transition from anomalous to normal diffusion depends only on the existence of a discontinuity in the derivative of $\tau(q)$ and hence on the discrete points in the $f(\alpha)$ spectrum. In the normal case where $\langle 1/W \rangle$ finite, i.e., $\sum_{i=1}^{N} 1/W_i \propto N \propto l^{-1}$ we find $\tau(1) = -1$ and $d_w = 2$.

Since the measure p_i of each segment describes the inverse of the transition rate we expect to find negative α in the normal and anomalous slow regime ($d_w \geq 2$). This follows since $\alpha_{\min} = \ln[\min(p_i)]/\ln l$, $\alpha_{\max} = \ln[\max(p_i)]/\ln l$, and $W_i \leq 1$. A transition to the ballistic regime ($d_w < 2$) occurs when $W_i > 1$; in this case we find positive α. In the following, we study two examples of systems exhibiting fractal transition probabilities and analyze their properties.

Consider the interval [0,1] with the following construction laws: each segment with the measure 1 is divided into 3 segments with the measures $1,R,1$ ($R > 1$); and each segment with the measure R^n ($n \neq 0$) is transformed by one iteration to R^{n+1}. As noted above, at each iteration the length of all segments l_i are equal (Fig. 1). We, therefore, construct *ad infinitum* using the above rules, a hierarchical structure of barriers which can be generated from an ultrametric tree with coordination number $z=3$. One finds for this system that, as $n\rightarrow\infty$,

$$\tau(q)=\min\left\{-1,\frac{\ln 1/R}{\ln 2}q\right\}. \tag{6}$$

Hence, the $f(\alpha)$ spectrum consists of two discrete points:

$$(f,\alpha)=\begin{cases}(1,0), & R>2\,,\\ \left(0,\dfrac{\ln 1/R}{\ln 2}\right), & 1<R\leq 2\end{cases} \tag{7}$$

and

$$d_w=\begin{cases}2, & 1<R\leq 2\,,\\ 1+\dfrac{\ln R}{\ln 2}, & R>2\,.\end{cases} \tag{8}$$

This sharp transition in the exponent was obtained earlier using a variety of methods.[12-17] The same transition holds when dealing with transition rates distributed randomly as $P(p_i)=p_i^{-\alpha}$ where $\alpha=1-\ln 2/\ln R$, as the order of p_i in the 1D chain is irrelevant.[16,17]

We next consider the case where the self similarity is perfect (see Fig. 2), and construct a fractal by iterating the process *ad infinitum*. We start with the interval [0,1] and use the following construction laws: The segment with the measure $p_i=1$ is divided at each generation to 1:R:1, and the segment with $p_i=R^n$ ($n\neq 0$) is divided into R^n, R^{n+1}, R^n. Using Eqs. (1) and (2), one obtains

$$\tau(q)=-\ln(2+R^q)/\ln 3\,, \tag{9}$$

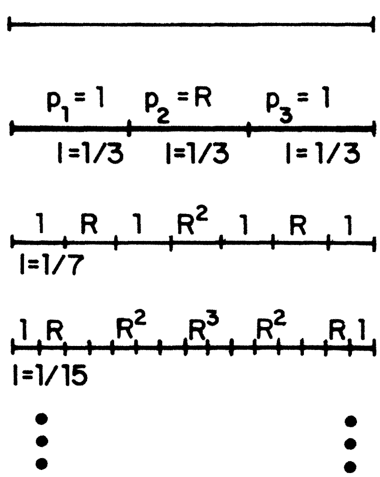

FIG. 1. The construction of the hierarchical set. Each segment with the measure $p=1$ is divided into $p_j=1$, $p_{j+1}=R$, $p_{j+2}=1$, and the segment with $p_k=R^n$ ($n\neq 0$) is multiplied by R to R^{n+1} at any generation. At each stage the lengths are equal. The measures are distributed at the infinite generation by $P(p_i)=p^{-\alpha}$, $\alpha=1-\ln 2/\ln R$. Note that the process is not completely self similar.

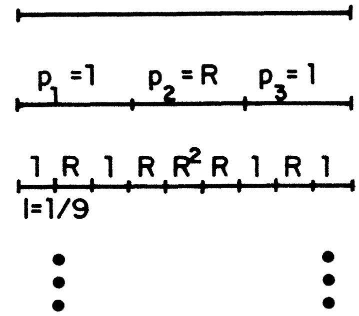

FIG. 2. The construction of the multifractal set. The division of the set is completely self similar. The measures are distributed after iteration by $P(p_j)=2^{n-j}\binom{n}{j}$, $P_j=R^j$ ($j=0,\ldots,n$).

and

$$f=\left[\alpha\ln\left(\frac{-2\alpha\ln 3}{\ln R+\alpha\ln 3}\right)\Big/\ln R\right]+\left[\ln\left(2-\frac{2\alpha\ln 3}{\ln R+\alpha\ln 3}\right)\Big/\ln 3\right]. \tag{10}$$

Using Eq. (5), we find, in contrast with the hierarchical case, Eq. (8), no transition,

$$d_w=1+\frac{\ln(2+R)}{\ln 3}\,, \tag{11}$$

which holds for all $R>0$. The motion is anomalously slow for $R>1$ and is anomalously "ballistic" for $R<1$, which corresponds to an effective-field operating on the particle.

Note that the f-α spectrum is symmetric for R and R^{-1} as seen in Fig. 3. The above results, Eqs. (9)–(11), were obtained for the deterministic multifractal structures (Fig. 2). The distribution of the transition rates in this structure is of a binomial form

$$P(p_i)=2^{n-j}\binom{n}{j}\,, \tag{12}$$

where $p_i=R^j$ ($j=0,\ldots,n$) at the nth generation.

Since the sum in Eq. (3) does not depend on the order of the segments (i.e., transition rates), we conclude that Eq. (11) holds for a random binomial distribution. In the continuum limit, one can approximate this distribution by a Gaussian.

The above rule $d_w=1-\tau(1)$ is asymptotic and valid for $t\rightarrow\infty$. When we deal with finite systems or finite times, we expect to find a crossover from an initial exponent, depending on the initial point of the random walker, to the asymptotic one. This crossover represents the lacunarity property of the fractal generated from the transition rates. One interesting property, however, is an effective different self-similarity, which the random walker sees when its initial starting point is exactly in the middle of the interval [0,1]. This diffusion case is represented by an effective subset of transition rates $p_1=R^{-1}$, $p_2=1$, $p_3=R^{-1}$, and hence, $\tau(q)=-\ln(2R^{-q}+1)/\ln 3$ and $d_w=1+\ln(2R^{-1}+1)/\ln 3$. For any finite but large iteration, this initial point represents the cell with the lowest transition rate,

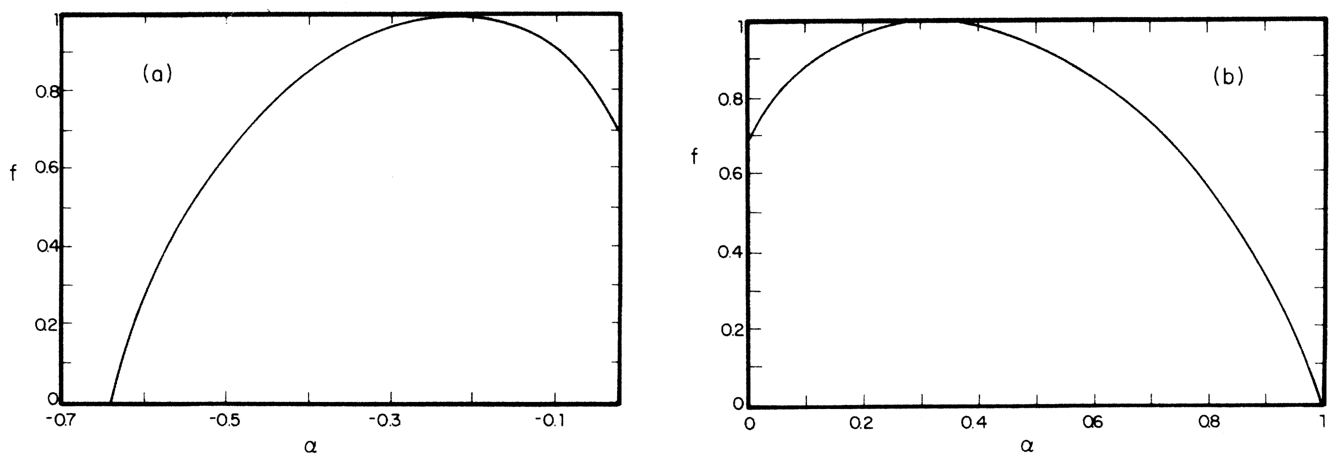

FIG. 3. (a) The $f(\alpha)$ spectrum of the multifractal set. In this figure, $R=2$, and as expected $f_{\max}=1$ which is the dimension of the subset. (b) The spectrum for $R=\frac{1}{3}$. The strange behavior (α positive) corresponds to an external ordering force which causes $d_w<2$.

approaching zero as n increases. These results were confirmed numerically to a high degree of accuracy.

To summarize, we extend the study of normalized measures of segments lying on a fractal set to nonnormalized measures. We study these cases in which $p_i>1$ which yields negative values for α. In these systems $\tau(1)$ is nonzero and represents the Hausdorff dimension of the system. We relate the Hausdorff dimension of the transition rates $\tau(1)$ in 1D systems to the dynamical properties such as the diffusion exponent.

We wish to acknowledge the help of Ofer Matan in the computer programming.

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